

UNIFORM REGULARITY FOR FREE-BOUNDARY NAVIER-STOKES EQUATIONS WITH SURFACE TENSION

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ABSTRACT. We study the zero-viscosity limit of free boundary Navier-Stokes equations with surface tension in \mathbb{R}^3 thus extending the work of Masmoudi and Rousset [1] to take surface tension into account. Due to the presence of boundary layers, we are unable to pass to the zero-viscosity limit in the usual Sobolev spaces. Indeed, as viscosity tends to zero, normal derivatives at the boundary should blow-up. To deal with this problem, we solve the free boundary problem in the so-called Sobolev co-normal spaces (after fixing the boundary via a coordinate transformation). We prove estimates which are uniform in the viscosity. And after inviscid limit process, we get the local existence of free-boundary Euler equation with surface tension. One of the main differences between this work and the work [1] is our use of time-derivative estimates and certain properties of the Dirichlet-Neumann operator. In a forthcoming work, we discuss how we can take the simultaneous limit of zero viscosity and surface tension [].

1. INTRODUCTION

The water-wave problem has been studied for several decades from several different points of view. First the local existence for free boundary problem of Navier-Stokes equation without surface tension was shown by Beale [5]. Allain [6] and Tani [7] proved local-existence for the free-boundary Navier-Stokes equation with surface tension in the case of two dimensions and three dimensions, respectively. Moreover, with surface tension, global regularity was also studied by Beale [8].

In the case where the fluid is assumed to be inviscid and irrotational, the problem can be reduced to the boundary. Recently, global regularity was achieved by S. Wu [11] and by Germain, Masmoudi and Shatah [12] for small data. In the general case (where the vorticity may be non-zero) local well-posedness was proven by a number of different authors first by Christodoulou and Lindblad [13] and Lindblad [14], then Coutand and Shkoller [16], Lannes [3], Shatah and Zheng [17], and Masmoudi and Rousset [1].

In this paper we consider the vanishing viscosity limit for the water wave problem when surface tension is taken into account. The inviscid limit problem of the free-surface Navier-Stokes equation *without* surface tension was studied by Masmoudi and Rousset in [1]. When surface tension is not taken into account, the boundary, h , has same regularity as the velocity, u , (say H^m). In the process of doing high order energy estimates, one loses half a derivative due to some commutators. That commutator comes from $D^m \nabla \varphi$, where φ is harmonic extension of h to the interior of the domain, which is $\frac{1}{2}$ more regular than h . The main idea of the paper [1] is to use Alinhac's good unknown which reduces the order of loss via a critical cancellation. A second important component of [1], which sets it apart from the rest of the works on the free-boundary Euler equations, is that the authors are forced to use spaces which only measure co-normal regularity. This is due to the presence of boundary layers which form during the process of sending the viscosity to zero. Indeed, because of the boundary layer, we expect that near the boundary u^ε behaves like

$$u^\varepsilon \sim u(t, x) + \sqrt{\varepsilon} U(t, y, z/\sqrt{\varepsilon}),$$

where u is the solution of the free-boundary Euler equation, U is a some profile, and y is 2-d horizontal variable. So, for high order Sobolev space, we cannot hope to get interval of time independent of ε , which is crucial to get strong compactness of solution sequences. Hence we consider a Sobolev co-normal space, in which we expect to maintain boundedness of the Lipschitz norm as well as boundedness of higher order co-normal derivatives on an interval of time independent of ε .

Now Let's consider the similar case with surface tension. We will still use Sobolev co-normal spaces like in [1] because boundary layers are still present. However, in this case the boundary is more regular so we will not need Alinhac's good unknown. Our main problem comes from the fact that the pressure term in the Euler equations becomes significantly less regular when surface tension is introduced. We thus encounter commutators with order $m + \frac{3}{2}$, which we cannot control. For this reason, we decided to do energy estimates

using space *and* time derivatives. This helps because time derivatives actually count for $3/2$ space derivatives on the boundary (this is deduced by studying the properties of the Dirichlet to Neumann mapping). Using this fact, we can derive local existence for a time interval independent to ε . Finally, we deduce the solution of free-boundary Euler equation (subject to surface tension) as $\varepsilon \rightarrow 0$, using a strong compactness argument.

1.1. The Free-boundary Navier-Stokes equations. We solve the incompressible free-boundary Navier-Stokes equations under the effect of gravity in an unbounded domain. Assume that above the free-surface of the fluid is a vacuum. The system we get is:

$$(1.1) \quad \partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u, \quad x \in \Omega_t, \quad t > 0,$$

$$(1.2) \quad \nabla \cdot u = 0, \quad x \in \Omega_t,$$

where Ω_t is domain, occupied by fluid. We write the fluid boundary as h , a function of x and y , so that

$$\Omega_t = \{x \in \mathbb{R}^3, \quad x_3 < h(t, x_1, x_2)\}.$$

Our first boundary condition is the moving boundary condition (or called kinematic boundary condition), which roughly says that the boundary moves with the fluid:

$$(1.3) \quad \partial_t h = u(t, x_1, x_2, h(t, x_1, x_2)) \cdot \mathbf{N}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

where $\mathbf{N} = (-\nabla h, 1)$. Our second boundary condition is the continuity of stress tensor on the boundary.

$$(1.4) \quad p^b \mathbf{n} - 2\varepsilon(Su)^b \mathbf{n} = gh - \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}, \quad x \in \partial\Omega_t,$$

where $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ and Su is the symmetric part of the gradient of u

$$Su = \frac{(\nabla u) + (\nabla u)^T}{2}$$

and η is the surface tension constant. In this paper, we consider system (1.1)-(1.4) and its vanishing viscosity limit $\varepsilon \rightarrow 0$. In this paper, we define $\mathbf{N} \doteq (-\nabla \varphi, 1)$ inside the domain, so it means normal vector $(-\nabla h, 1)$ on the free-boundary. And \mathbf{n} is normalized vector $\frac{\mathbf{N}}{|\mathbf{N}|}$.

1.2. Parametrization to a Fixed domain. The first step is to rewrite the problem on the fixed domain $S = \{(x, y, z) | z < 0\}$. This can be done by a diffeomorphism $\Phi(t, \cdot)$,

$$(1.5) \quad \Phi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \rightarrow \Omega_t, \\ x = (y, z) \mapsto (y, \varphi(t, y, z)).$$

We use function v and q to denote the velocity and pressure on the fixed domain S .

$$(1.6) \quad v(t, x) = u(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x)).$$

There are several different choices for Φ and we have to decide which one is optimal for our purposes. We will need to find $\varphi(t, \cdot)$ so that $\Phi(t, \cdot)$ is a diffeomorphism (Surely, $\partial_z \varphi \geq 0$, because it is diffeomorphism). One easy option is to set $\varphi(t, y, z) = z + \eta(t, y)$. But it is more useful to take a function Φ which is actually *more* regular than h . If one thinks about using a harmonic extension, we see that it is possible for Φ to be $\frac{1}{2}$ of a derivative more regular than h . We take a smoothing diffeomorphism as was done in [1]:

$$(1.7) \quad \varphi(t, y, z) = Az + \eta(t, y, z).$$

To ensure that $\Phi(0, \cdot)$ is a diffeomorphism, A should be chosen so that

$$(1.8) \quad \partial_z \varphi(0, y, z) \geq 1, \quad \forall x \in S$$

and η is given by the extension of h to the domain S , defined by

$$(1.9) \quad \hat{\eta}(\xi, z) = \chi(z\xi)\hat{h}(\xi),$$

where χ is a smooth, compactly supported function which is 1 on the unit ball $B(0, 1)$. This smoothing diffeomorphism was used in [3],[4], and also in [1]. For this extension, the regularity of φ (of course η also) is $\frac{1}{2}$ of a derivative better than h . This will be explained in the next section in more detail.

We want to rewrite equations (1.1)-(1.4) on the moving domain Ω_t as equations on the fixed domain S . Using change of variables, we get

$$(1.10) \quad (\partial_i u)(t, y, \phi) = (\partial_i v - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z v)(t, y, z), \quad i = t, 1, 2,$$

$$(\partial_3 u)(t, y, \phi) = (\frac{1}{\partial_z \varphi} \partial_z v)(t, y, z).$$

So we define the following operator like in [1]:

$$(1.11) \quad \partial_i^\varphi = \partial_i - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z, \quad i = t, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_z \varphi} \partial_z,$$

This definition implies that $\partial_i u \circ \Phi = \partial_i^\varphi v$, $i = t, 1, 2, 3$.

Hence our equations in S are,

$$(1.12) \quad \partial_t^\varphi v + v \cdot \nabla^\varphi v + \nabla^\varphi q = \varepsilon \Delta^\varphi v, \quad x \in S, \quad t > 0,$$

$$(1.13) \quad \nabla^\varphi \cdot v = 0, \quad x \in S,$$

$$(1.14) \quad \partial_t h = v \left(t, x_1, x_2, h(t, x_1, x_2) \right) \cdot N, \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$(1.15) \quad q^b \mathbf{n} - 2\varepsilon (S^\varphi v)^b \mathbf{n} = gh - \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}, \quad \text{on } \partial S.$$

1.3. Functional Framework and Notations. We introduce co-normal space and some function spaces that are tailored to our problem. First we define Sobolev co-normal derivatives as:

$$(1.16) \quad Z_1 = \partial_x, \quad Z_2 = \partial_y, \quad Z_3 = \frac{z}{1-z} \partial_z, \quad Z^\alpha = Z^{(\alpha_1, \alpha_2, \alpha_3)}.$$

From now on, we use the following symbol:

$$Z^m \doteq \partial_t^k Z^\alpha \quad \text{for some } (k + |\alpha| = m).$$

There are many combinations that satisfy $k + |\alpha| = m$, but we will sum all the cases later, so we don't have to distinguish each case. About norms in co-normal spaces, as usual,

$$(1.17) \quad \|f\|_{H_{co}^s}^2 \doteq \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^2}^2, \quad \|f\|_{W_{co}^{s,\infty}} \doteq \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^\infty}.$$

In this paper we abbreviate the notation as $|\cdot|_s = |\cdot|_{H^s}$, $\|\cdot\|_s = \|\cdot\|_{H_{co}^s}$ and $\|\cdot\| = \|\cdot\|_{L^2}$. And similarly $|\cdot|_{s,\infty} = \|\cdot\|_{W_{co}^{s,\infty}}$ and $\|\cdot\|_{s,\infty} = \|\cdot\|_{W_{co}^{s,\infty}}$. Now we define function spaces $X^{m,s}$ and $Y^{m,s}$ as follows:

Definition 1.1. We define the space $X^{m,s}$ and $Y^{m,s}$ by the norms

$$(1.18) \quad |h|_{X^{m,s}}^2 \doteq \sum_{(k,\alpha), k+|\alpha| \leq m} |\partial_t^k D^\alpha h|_s^2,$$

$$(1.19) \quad \|u\|_{X^{m,s}}^2 \doteq \sum_{(k,\alpha), k+|\alpha| \leq m} \|\partial_t^k Z^\alpha u\|_s^2,$$

$$(1.20) \quad |h|_{Y^{m,s}} \doteq \sum_{(k,\alpha), k+|\alpha| \leq m} |\partial_t^k D^\alpha h|_{s,\infty},$$

$$(1.21) \quad \|u\|_{Y^{m,s}} \doteq \sum_{(k,\alpha), k+|\alpha| \leq m} \|\partial_t^k Z^\alpha u\|_{s,\infty},$$

where D^α means horizontal derivatives.

1.4. Main Result. For convenience, we will use the following convention regarding co-normal spaces with fractional indices:

$$\frac{m}{2} \doteq \begin{cases} \frac{m}{2}, & m \text{ even} \\ \frac{m-1}{2}, & m \text{ odd.} \end{cases}$$

Theorem 1.2. *For fixed sufficiently large $m \geq 9$, let initial data be given so that*

$$(1.22) \quad \begin{aligned} I_m(0) &\doteq \sum_{k=0}^m \left(\|(\partial_t^k v)(0)\|_{H_{co}^{m-k}} + |(\partial_t^k h)(0)|_{H^{m+1-k}} \right) + |\partial_t^{m-1} \nabla h(0)|_{\frac{3}{2}} \\ &\quad + \sum_{k=0}^{m-3} \|(\partial_t^k \nabla v)(0)\|_{H_{co}^{m-3-k}} + \sum_{k=0}^{m/2} \|(\partial_t^k \nabla v)(0)\|_{W_{co}^{\frac{m}{2}-k, \infty}} \leq R \end{aligned}$$

and satisfy compatibility conditions

$$(1.23) \quad \Pi \left(S^\varphi \partial_t^j v^\varepsilon(0) \right) \mathbf{n} = 0, \quad 0 \leq j \leq m, \quad \Pi \doteq \mathbf{I} - \mathbf{n} \otimes \mathbf{n}.$$

Then for $\forall \varepsilon \in (0, 1]$, there exist $T > 0$ (independent to ε), and some $C > 0$, such that there exist a unique solution $(v^\varepsilon, h^\varepsilon)$ on $[0, T]$, and the following energy estimate hold.

$$(1.24) \quad \begin{aligned} \sup_{[0, T]} \left(\|v\|_{X^{m-1,1}}^2 + |h|_{X^{m-1,2}}^2 + \|\partial_z v\|_{X^{m-3,0}}^2 + \|\partial_z v\|_{Y^{\frac{m}{2},0}}^2 \right) \\ + \|\partial_t^m v\|_{L^4 L^2}^2 + (1 + \|\partial_t^m \nabla h\|_{L^4 L^2}^2) + \|\partial_z v\|_{L^4 X^{m-1,0}}^2 < C, \end{aligned}$$

$$(1.25) \quad \varepsilon \int_0^T \int_0^s \|\nabla \partial_t^m v(\tau)\|_{L^2(S)}^2 d\tau ds + \varepsilon \int_0^T \left(\|\nabla v\|_{X^{m-1,1}}^2 + \|\nabla \partial_z v\|_{X^{m-3,0}}^2 \right) < C.$$

Next we get a unique solution of free boundary Euler equations with surface tension via the zero-viscosity limit.

Theorem 1.3. *Let us assume that*

$$(1.26) \quad \lim_{\varepsilon \rightarrow 0} \left(\|v_0^\varepsilon - v_0\|_{L^2(S)} + \|h_0^\varepsilon - h_0\|_{H^1(\partial S)} \right) = 0,$$

where $(v_0^\varepsilon, h_0^\varepsilon)$ and (v_0, h_0) satisfy the assumptions of Theorem 1.2. Then there exist (v, h) satisfying

$$(1.27) \quad v \in L^\infty([0, T], H_{co}^m(S)), \quad \partial_z v \in L^\infty([0, T], H_{co}^{m-3}(S)), \quad h \in L^\infty([0, T], H_{co}^{m+1}(\mathbb{R}^2))$$

and

$$(1.28) \quad \lim_{\varepsilon \rightarrow 0} \sup_{[0, T]} \left(\|v^\varepsilon - v\|_{L^2(S)} + \|v^\varepsilon - v\|_{L^\infty(S)} + \|h^\varepsilon - h\|_{H^1(\mathbb{R}^2)} + \|h^\varepsilon - h\|_{W^{1,\infty}(\mathbb{R}^2)} \right) = 0.$$

Moreover, (v, h) is the unique solution of free boundary Euler equation,

$$(1.29) \quad \partial_t^\varphi v + (v \cdot \nabla^\varphi) v + \nabla^\varphi q = 0, \quad \nabla^\varphi \cdot v = 0$$

with free boudnary

$$\partial_t h = v^b \cdot N$$

and

$$q = gh - \eta \nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right)$$

on the boundary.

1.5. Scheme of the Proof. We briefly explain main idea of this paper in several steps.

Remark 1.4. In this paper $\Lambda(\cdot, \cdot)$ denotes an increasing continuous function in all its arguments and $\Lambda_0 = \Lambda(\frac{1}{c_0})$. Both may vary from line to line.

1.5.1. *Energy estimate of v and h .* First let us apply $Z_x^m = \partial_x^{\alpha_x} \partial_y^{\alpha_y} \left(\frac{z}{1-z} \partial_z \right)^{\alpha_z}$, then our estimate looks like,

$$(1.30) \quad \begin{aligned} E_0 &\doteq \|v\|_{H_{co}^m}^2 + |h|_{H^{m+1}}^2 + \varepsilon \int_0^t \|\nabla v\|_{H_{co}^m}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_0(s) + \|\nabla v\|_{H_{co}^{m-1}}^2 + |h|_{H^{m+\frac{3}{2}}}^2 \right) ds, \end{aligned}$$

where C_0 is some terms depending on the initial data, R contains E_0 and some low order L^∞ -type terms. The problem is that $|h|_{H^{m+\frac{3}{2}}}$, on the right hand side cannot be controlled by E_0 . This term comes from the pressure estimates involving the surface tension term. To estimate $|h|_{H^{m+\frac{3}{2}}}$, we use Dirichlet-Neumann estimates and time-derivatives and a special decomposition of the pressure term. Indeed, we decompose the pressure into several pieces. One piece, q^S (which is the pressure due to the surface tension term) is such that q^S solves $\Delta q^S = 0$, $q^S|_b = -\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}$. Then, Then, we estimate Navier-Stokes equation on the boundary and the kinematic boundary condition,

$$h_{tt} = v_t^b \cdot N + v^b \cdot N_t.$$

Roughly,

$$(1.31) \quad h_{tt} = -(\nabla q^S)^b \cdot N + v^b \cdot N_t + (\text{commutators}).$$

Since $(q^S)^b \sim \Delta h$, we can get $h_{tt} \sim \nabla \Delta h$, so *heuristically*, $\partial_t h \sim \partial_x^{\frac{3}{2}} h$. However this relation is valid in the sense of L^2 in time. From proposition 7.2, we will see that roughly,

$$|Z_x^{|\alpha|} \partial_t^\beta \nabla h|_{L^2 H^{\frac{3}{2}}}^2 \sim |Z_x^{|\alpha|} \partial_t^\beta \nabla \partial_t h|_{L^2 L^2}^2 + \bar{\theta} |Z_x^{|\alpha|} \partial_t^\beta \nabla \partial_t h(t)|_{L^2}^2, \quad \text{for sufficiently small } \bar{\theta}.$$

$|Z_x^{|\alpha|} \partial_t^\beta \nabla \partial_t h(t)|_{L^2}^2$ part will be absorbed by energies of left hand sides of the next step energy which is gained by applying $Z_x^{|\alpha|-1} \partial_t^{\beta+1}$.

$$(1.32) \quad \begin{aligned} E_1(t) &\doteq \|\partial_t v\|_{H_{co}^{m-1}}^2 + |\partial_t h|_{H^m}^2 + \varepsilon \int_0^t \|\nabla \partial_t v\|_{H_{co}^{m-1}}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_1(s) + \|\nabla \partial_t v\|_{H_{co}^{m-2}}^2 + |\partial_t h|_{H^{m+\frac{1}{2}}}^2 \right) ds, \end{aligned}$$

where in this case, R contains E_0 and E_1 and some low order L^∞ -type terms. Similar situation will happen and we need the energy of $E_2(t)$. And is repeated until the last step.

$$(1.33) \quad \begin{aligned} E_2(t) &\doteq \|\partial_t^2 v\|_{H_{co}^{m-2}}^2 + |\partial_t^2 h|_{H^{m-1}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^2 v\|_{H_{co}^{m-2}}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_2(s) + \|\nabla \partial_t^2 v\|_{H_{co}^{m-3}}^2 + |\partial_t^2 h|_{H^{m-\frac{1}{2}}}^2 \right) ds, \end{aligned}$$

⋮

$$(1.34) \quad \begin{aligned} E_k &\doteq \|\partial_t^k v\|_{H_{co}^{m-k}}^2 + |\partial_t^k h|_{H^{m-k+1}}^2 + \varepsilon \int_0^t \|\nabla \partial_t^k v\|_{H_{co}^{m-k}}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_k(s) + \|\nabla \partial_t^k v\|_{H_{co}^{m-k-1}}^2 + |\partial_t^k h|_{H^{m-k+\frac{3}{2}}}^2 \right) ds, \quad 1 \leq k \leq m-1, \end{aligned}$$

⋮

$$(1.35) \quad \begin{aligned} E_{m-1} &\doteq \|\partial_t^{m-1} v\|_{H_{co}^1}^2 + |\partial_t^{m-1} h|_{H^2}^2 + \varepsilon \int_0^t \|\nabla \partial_t^{m-1} v\|_{H_{co}^1}^2 \\ &\leq C_0 + \Lambda(R) \int_0^t \left(E_{m-1}(s) + \|\nabla \partial_t^{m-1} v\|_{H_{co}^1}^2 + |\partial_t^{m-1} h|_{H^{\frac{5}{2}}}^2 \right) ds. \end{aligned}$$

If we sum the above m estimates, then $E_0 + E_1 + \dots + E_{m-2} + E_{m-1}$ controls every high order term of h except $|\partial_t^{m-1}h|_{L^2 H^{\frac{5}{2}}}^2$ and θ .

1.5.2. *Energy estimate for all-time derivatives.* In the last step, the last step energy is

$$E^m \doteq \|\partial_t^m v\|_{L^2}^2 + |\partial_t^m h|_{H^1}^2 + \varepsilon \int_0^t \|\nabla \partial_t^m v\|_{L^2}^2.$$

We apply ∂_t^m to the equation. We claim that in the step, the worst commutator $|\partial_t^m h|_{H^{3/2}}$ actually *does not* appear. Moreover we don't have $\|\partial_t^m q\|_{L^2}$ from elliptic estimate. $\|\partial_t^{m-1} \nabla q\|_{L^2}$ is maximal in terms of time derivatives. To see this let us investigate where the bad commutator comes from. The worst commutators those cannot be controlled come from two parts, (5.14) and (5.37).

The **First part** is $\partial_t^m q$. High-order commutators come from the commutator between ∂_t^m and $\partial_x^\varphi v$ and their product with $\partial_t^m q$. In other words:

$$(1.36) \quad \int_S \partial_t^m v \cdot \nabla^\varphi \partial_t^m q^S \sim \int_S (\nabla^\varphi \cdot \partial_t^m v) \partial_t^m q^S.$$

We use divergence free condition for v and integration by part in time and space generate

$$\int_0^t \int_{\partial S} \partial_t^m q^b m \partial_t \mathbf{N} \partial_t^{m-1} v^b$$

The **second part** is the coupling of surface tension and commutators from kinematic boundary term. i.e

$$\int_0^t \int_{\partial S} \partial_t^m \left(\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) m \partial_t \mathbf{N} \partial_t^{m-1} v^b.$$

In the first part, problematic highest order part of q^b is surface tension part, $\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}$. This cancels with second part. Other terms are controllable, except $\partial_t^m q$. From elliptic estimate for pressure, we cannot use $L^2 L^2$ control for $\int_S \partial_t^m q \partial_t^{m-1} \nabla v$. Instead we apply integration by part in time and space to change ∂_t and ∇ . This generate

$$\|\partial_t^{m-1} \nabla q(t)\|_{L^2} \sim |\partial_t^{m-1} \nabla h(t)|_{3/2}$$

on the right hand side without time integral. To control $|\partial_t^{m-1} \nabla h(t)|_{3/2}$, we need L^2 in time as explained before, $\partial_t h \sim \partial_x^{3/2} h$ in $L^2 L^2$. Hence we square this estimate and integrate in time again to make this as L^2 in time. Therefore we get L_t^4 type energy in the last step, $Z^m = \partial_t^m$. Considering all these steps, we now sum all $m+1$ energy estimates, $Z^m = \partial_x^m, \partial_x^{m-1} \partial_t, \dots, \partial_x^1 \partial_t^{m-1}, \partial_t^m$. Finally, except for estimating the ∇v terms, we are able to close the energy estimates if we are only concerned about the regularity of h . Meanwhile, from the definition of conormal estimate our energy cannot control $\|Z^{m-1} \nabla v\|_{L^2}$. This part is explained in the following. Note that this part has nothing to do with all-times energy estimate, $Z^m = \partial_t^m$.

1.5.3. *L^2 -type normal estimate.* Our commutators contain $\|\partial_z v\|_{X_{m-1,0}}$, which cannot be controlled, since our v has only co-normal regularity. As in [1], we make energy estimates of S_n , the tangential part of $S^\varphi v \mathbf{n}$, because $\partial_z v \sim S_n$ (instead of ∂v , it is suffice to estimate S_n). However D^2 destroy divergence free structure of velocity fields, so we cannot reduce the order of pressure. Considering pressure estimate, we get the estimate of S_n :

$$(1.37) \quad \|S_n\|_{X_{m-3,0}}^2 + \varepsilon \int_0^t \|\nabla S_n\|_{X_{m-3,0}}^2 \leq C_0 + \Lambda(R) \int_0^t \left(E(s) + \|S_n(s)\|_{X_{m-3,0}} \right) ds.$$

Unfortunately we failed to get $m-1$ estimate for $\partial_z v$. But this estimate can be used to control L^∞ type estimate.

1.5.4. *L^∞ -type normal estimate.* Next, we do L^∞ -type estimates, which is included in R above. We estimate S_n instead of $\partial_z v$. The main difficulty is the commutator between Z_3 and Laplacian. We consider a thin layer near the boundary and reparameterize so that $\partial_z^\varphi \partial_z^\varphi$ look like ∂_{zz} . And then, we change the advection term as

$$\partial_t \rho + (w_y(t, y, 0), zw_3(t, y, 0)) \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = l.o.t.$$

We do not apply a simple maximum principle for convection-diffusion equations. We apply Duhamel's principle using Green's evolution kernel. Then we can conclude

$$\|\partial_z v\|_{Y^{\frac{m}{2},0}}^2 \lesssim \|\partial_v(0)\|_{Y^{\frac{m}{2},0}}^2 + \Lambda(R) \int_0^t \varepsilon \|\partial_{zz} v\|_{X^{m-3,0}}^2.$$

1.5.5. Vorticity estimate. We still have to control $\|\partial_z v\|_{X^{m-1,0}}$. We use good idea of [1], vorticity estimate, which is equivalent to $\partial_z v$. The biggest merit of vorticity is that taking curl removes pressure. This makes us possible to get $m-1$ order estimate. However we are considering general vorticity data on the free-boundary, hence we cannot get L_t^∞ energy estimate. In [1], L_t^4 estimate for vorticity was introduced, based on the heat equation with general boundary data and paradifferential calculus.

1.5.6. Uniform Existence and Uniqueness. At this point we have made all the necessary estimates to close the main energy estimate (1.30). In particular, the right hand side of the energy estimate is independent of ε , provided the energy remains bounded. So, using the preliminary existence result of A.Tani [7] and strong compactness arguments, we get local existence. For uniqueness, it is suffice to do L^2 -estimate for the difference of two solutions and we conclude by Gronwall's inequality.

1.6. Comparing the problem with and without surface tension. Surface tension is, overall, a regularizing force in the water wave problem; however, it introduces several (perhaps unexpected) difficulties. Here we want to elaborate upon the differences between the paper of Masmoudi-Rousset [1] (the case where no there is no surface tension) and our result (where surface tension is taken into account. In terms of the basic functional framework, both works use Sobolev co-normal spaces due to the presence of boundary layers. However, there are big differences between these two works. First, let us look at a scheme of [1] (no surface tension case). When we have no surface tension, m -order energy estimate contains $|h|_m$. The main problem which the authors faced in [1] is the presence of certain high order commutators. To get around this problem, the authors made use of Alinhac's good unknown which allowed them to close the energy estimates. They use the good unknowns: $V^\alpha = Z^\alpha v - \partial_z^\varphi v Z^\alpha \eta$ and $Q^\alpha = Z^\alpha q - \partial_z^\varphi q Z^\alpha \eta$, because, with this new variable, the bad commutator $Z^\alpha N$ is disappears. The second major problem is in estimating one normal derivative near the boundary its optimal regularity is $m-2$, not $m-1$, because of the regularity of h . ($|S_n|_{m-1}$ estimate require $|h|_{m+\frac{1}{2}}$ which show lack of $\frac{1}{2}$ regularity.) This is why, [1] requires quiet complicate analysis (need to estimate $\|w\|_{L^4 H_{co}^{m-1}}$) to control $\partial_z v$ of commutator. Both of these problems would disappear if the surface were more regular.

Meanwhile, in the surface tension case (this paper), the m^{th} -order energy estimate has $|h|_{m+1}$ in its energy. So now one doesn't need Alinhac's good unknown. Nevertheless, we also lack $\frac{1}{2}$ order ($|h|_{m+\frac{3}{2}}$ appears in the commutators) because of the pressure. Because we use co-normal spaces,

$$\int_S Z^m v \cdot \nabla^\varphi Z^m q$$

make high order commutator about pressure q in S (which vanishes in case of standard Sobolev space derivatives D^m , by divergence free condition). Since $q^b \sim \Delta h$, $q \sim \partial_x^{\frac{3}{2}} h$. As mentioned in above scheme, it is bounded by taking time derivatives. The crucial point is that when we only take time derivatives of the equation the worst commutator does not show up. In our case (as opposed to [1]), normal derivatives are easier to deal with, since $\|S_n\|$ has optimal $m-1$ order regularity, by which we can close energy estimate.

Regarding L^∞ estimates for S_n , [1] requires $\varepsilon \|\partial_{zz} v\|_{L^\infty}$. But we do not need $\varepsilon \|\partial_{zz} v\|_{Y^{k,0}}$. This is because, $\varepsilon \|\partial_{zz} v\|_{L^\infty}$ appears by Alinhac's unknown which include $\partial_z v Z^\alpha \eta$.

2. BASIC PROPOSITIONS

2.1. Basic propositions. We construct some proposition to estimate commutators.

Proposition 2.1. *For $m \in 2\mathbb{N}$, we get the following estimates.*

$$(2.1) \quad \|Z^m(uv)\| \lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m,0}} \|u\|_{Y^{\frac{m}{2},0}},$$

$$(2.2) \quad \|[Z^m, u]v\| \lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{\frac{m}{2},0}},$$

$$(2.3) \quad \|[Z^m, u, v]\| \lesssim \|u\|_{X^{m-1,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{\frac{m}{2},0}}.$$

Proof. We cannot use general Leibnitz Rule since $Z_3 = \frac{z}{1-z}\partial_z$, but every order of derivatives of $\frac{z}{1-z}$ is uniformly bounded, so we can use similar estimate if we use \lesssim instead of \leq .

$$(2.4) \quad \begin{aligned} \|Z^m(uv)\| &\leq \sum_{(\beta,\gamma), |\beta|+|\gamma|\leq m} \|Z^\beta u Z^\gamma v\| = \sum_{|\beta|\geq |\gamma|} \|Z^\beta u Z^\gamma v\| + \sum_{|\beta|\leq |\gamma|} \|Z^\beta u Z^\gamma v\| \\ &\leq \sum_{|\beta|\geq |\gamma|} \|Z^\beta u\| |Z^\gamma v|_{L^\infty} + \sum_{|\beta|\leq |\gamma|} |Z^\beta u|_{L^\infty} \|Z^\gamma v\| \lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m,0}} \|u\|_{Y^{\frac{m}{2},0}}, \end{aligned}$$

$$(2.5) \quad \| [Z^m, u] v \| \lesssim \sum_{|\beta|+|\gamma|=m, \beta \neq 0} \|Z^\beta u Z^\gamma v\| \lesssim \|u\|_{X^{m,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{\frac{m}{2},0}},$$

$$(2.6) \quad \| [Z^m, u] v \| \lesssim \sum_{|\beta|+|\gamma|=m, \beta \neq 0, \gamma \neq 0} \|Z^\beta u Z^\gamma v\| \lesssim \|u\|_{X^{m-1,0}} \|v\|_{Y^{\frac{m}{2},0}} + \|v\|_{X^{m-1,0}} \|u\|_{Y^{\frac{m}{2},0}}.$$

□

Remark 2.2. The idea is that for each bilinear term, we put the L^2 norm on the term with higher derivatives and the L^∞ norm to low order term. In co-normal derivatives, there is no proper notion of fractional derivatives, so $Z_3^{1/2}$ does not make sense. We deal when m is even, so that $\frac{m}{2}$ is also a integer, but our result also work for odd m , because it suffices to give $\frac{m-1}{2}$ orders to L^∞ and $\frac{m+1}{2}$ orders to L^2 . So in this paper, $\frac{m}{2}$ means, integer $\frac{m-1}{2}$, when m is odd. But for convenience, we abuse notation. It does not make any problem because, if we pick m as sufficiently large, these L^∞ type terms will be controlled by energy which has order m .

The followings are anisotropic embeddings and trace properties in co-normal spaces.

Proposition 2.3. 1) $s_1 \geq 0, s_2 \geq 0$ such that $s_1 + s_2 > 2$ and u such that $u \in H_{tan}^{s_1}, \partial_z u \in H_{tan}^{s_2}$, we have the anisotropic Sobolev embedding:

$$(2.7) \quad |u|_{L^\infty}^2 \lesssim \|\partial_z u\|_{H_{tan}^{s_2}} \|u\|_{H_{tan}^{s_1}}.$$

2) For $u \in H^1(S)$, we have the trace estimate :

$$(2.8) \quad |u(\cdot, 0)|_{H^s(\mathbb{R}^2)} \lesssim \|\partial_z u\|_{H_{tan}^{s_2}}^{\frac{1}{2}} \|u\|_{H_{tan}^{s_1}}^{\frac{1}{2}}$$

with $s_1 + s_2 = 2s \geq 0$

Proof. see [1].

□

2.2. Estimate of η . As explained before, the reason we choose a smoothing diffeomorphism is that η is $\frac{1}{2}$ derivative more regular than h . This fact is crucial later, because this term can accommodate an extra $\frac{1}{2}$ derivative in bilinear estimates. For example, in the pressure estimates: i.e

$$\int_S (\nabla \varphi) q \leq \|\nabla \varphi\|_{\frac{1}{2}} \|q\|_{-\frac{1}{2}} \sim \|\nabla h\|_{L^2} \|q\|_{-\frac{1}{2}}.$$

We define diffeomorphism so that at initial time, $\partial_z \varphi(0, y, z) \geq 1$. $\partial_z \varphi$ should be positive during our estimates, so our estimate is valid during on $[0, T^\varepsilon]$ such that

$$(2.9) \quad \partial_z \varphi(t, y, z) \geq c_0, \quad \forall t \in [0, T^\varepsilon]$$

for some c_0 .

Proposition 2.4. For η , we obtain the following estimates.

$$(2.10) \quad |\nabla \eta|_{H^s(S)} \leq C_s |h|_{s+\frac{1}{2}},$$

$$(2.11) \quad |\nabla \eta|_{X^{m,0}} \lesssim C_s |h|_{X^{m,\frac{1}{2}}}.$$

Moreover, for L^∞ type, we get

$$(2.12) \quad \forall s \in \mathbb{N}, \quad |\eta|_{W^{s,\infty}} \leq C_s |h|_{s,\infty},$$

$$(2.13) \quad \forall s \in \mathbb{N}, \quad |\eta|_{Y^{m,0}} \lesssim C_s |h|_{Y^{m,0}}.$$

Proof. The first inequality is from [1], and $|\nabla \partial_t^k \eta|_{H^s(S)} \leq C_s |\partial_t^k h|_{s+\frac{1}{2}}$ is also trivial by definition of η . So, by summing all cases, we get the second inequality. For L^∞ type estimates, the third inequality is from [1], and the last inequality is also trivial by definition of η . \square

The following lemma is useful to estimate, since we will see many terms like $\frac{u}{\partial_z \varphi}$.

Lemma 2.5. *We have the following estimate.*

$$(2.14) \quad \left\| Z^m \frac{u}{\partial_z \varphi} \right\| \lesssim \Lambda \left(\frac{1}{c_0}, \|u\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|u\|_{X^{m,0}} + |h|_{X^{m,\frac{1}{2}}} \right).$$

Proof. $F(x) = x/(A+x)$ is a smooth function of which all order derivatives are bounded when $A+x \geq c_0 > 0$. So,

$$(2.15) \quad \begin{aligned} \left\| Z^m \frac{u}{\partial_z \varphi} \right\| &= \left\| Z^m \left(\frac{u}{A} - \frac{u}{A} F(\partial_z \eta) \right) \right\| \lesssim \|u\|_{X^{m,0}} + \|Z^m(uF(\partial_z \eta))\| \\ &\lesssim \|u\|_{X^{m,0}} + \|u\|_{X^{m,0}} \|F(\partial_z \eta)\|_{Y^{\frac{m}{2},0}} + \|u\|_{Y^{\frac{m}{2},0}} \|F(\partial_z \eta)\|_{X^{m,0}}. \end{aligned}$$

Meanwhile,

$$(2.16) \quad \begin{aligned} \|F(\partial_z \eta)\|_{X^{m,0}} &\lesssim \Lambda \left(\frac{1}{c_0}, |\nabla \eta|_{Y^{\frac{m}{2},0}} \right) |\partial_z \eta|_{X^{m,0}} \lesssim \Lambda \left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} \right) |h|_{X^{m,\frac{1}{2}}} \\ \|F(\partial_z \eta)\|_{Y^{\frac{m}{2},0}} &\lesssim \Lambda \left(\frac{1}{c_0}, |\nabla \eta|_{Y^{\frac{m}{2},0}} \right) \lesssim \Lambda \left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} \right). \end{aligned}$$

Hence, we get the result. \square

2.3. Dissipation term control. We comment a kind of version of Korn's inequality for S^φ . From [1], we have

Proposition 2.6. *If $\partial_z \varphi \geq c_0$, $\|\nabla \varphi\|_{L^\infty} + \|\nabla^2 \varphi\|_{L^\infty} \leq \frac{1}{c_0}$ for some $c_0 > 0$, then there exists $\Lambda_0 = \Lambda_0 > 0$ such that for every $v \in H^1(S)$, we have*

$$(2.17) \quad \|\nabla v\|_{L^2(S)}^2 \leq \Lambda_0 \left(\int_S |S^\varphi v|^2 dV_t + \|v\|^2 \right),$$

where

$$S^\varphi v = \frac{1}{2} \left(\nabla^\varphi v + (\nabla^\varphi v)^T \right).$$

Also, we have the following estimate.

$$(2.18) \quad \|\nabla v\|_{X^{m,0}}^2 \leq \Lambda_0 \left(\int_S |S^\varphi v|_{X^{m,0}}^2 + \|v\|_{X^{m,0}}^2 \right).$$

Proof. See [1] for the first estimate. For second inequality, we apply same estimate for $\partial^k Z^\alpha v$. \square

3. EQUATIONS OF $(Z^m v, Z^m h, Z^m q)$

3.1. Commutator estimate.

Proposition 3.1. *For $i = t, 1, 2, 3$, let us define*

$$(3.1) \quad Z^m(\partial_i^\varphi f) = \partial_i^\varphi(Z^m f) + C_i^m(f).$$

Then we have,

$$(3.2) \quad \|C_i^m(f)\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla f\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|\nabla f\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right).$$

Proof. For $i = t, 1, 2$

$$(3.3) \quad \begin{aligned} Z^m \left(\partial_i f - \frac{\partial_i \varphi}{\partial_z \varphi} \partial_z f \right) &= \partial_i \left(Z^m f \right) - Z^m \left(\frac{\partial_i \varphi}{\partial_z \varphi} \partial_i^\varphi f \right) \\ &= \partial_i \left(Z^m f \right) - \left(\left[Z^m, \frac{\partial_i \varphi}{\partial_z \varphi} \right] \partial_z f + \left(Z^m \frac{\partial_i \varphi}{\partial_z \varphi} \right) \partial_z f + \frac{\partial_i \varphi}{\partial_z \varphi} \left(Z^m \partial_z f \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \partial_i^\varphi(Z^m f) - \left(\left[Z^m, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f \right] + \left(Z^m \frac{\partial_i \varphi}{\partial_z \varphi} \right) \partial_z f + \frac{\partial_i \varphi}{\partial_z \varphi} [Z^m, \partial_z] f \right) \\
&= \partial_i^\varphi(Z^m f) + C_i^m(f).
\end{aligned}$$

Now we estimate three terms using above propositions and lemmas.

$$\begin{aligned}
(3.4) \quad \left\| \left[Z^m, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f \right] \right\| &\lesssim \left\| \frac{\partial_i \varphi}{\partial_z \varphi} \right\|_{X^{m-1,0}} \|\partial_z f\|_{Y^{\frac{m}{2},0}} + \left\| \frac{\partial_i \varphi}{\partial_z \varphi} \right\|_{Y^{\frac{m}{2},0}} \|\partial_z f\|_{X^{m-1,0}} \\
&\lesssim \Lambda \left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} + \|\partial_z f\|_{Y^{\frac{m}{2},0}} \right) \left(|h|_{X^{m-1,\frac{1}{2}}} + \|\partial_z f\|_{X^{m-1,0}} \right),
\end{aligned}$$

$$\begin{aligned}
(3.5) \quad \left\| \left(Z^m \frac{\partial_i \varphi}{\partial_z \varphi} \right) \partial_z f \right\| &\lesssim |\partial_z f|_{L^\infty} \left\| Z^m \frac{\partial_i \varphi}{\partial_z \varphi} \right\| \\
&\lesssim \Lambda \left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} + |\partial_z f|_{L^\infty} \right) |h|_{X^{m,\frac{1}{2}}},
\end{aligned}$$

$$(3.6) \quad \left\| \frac{\partial_i \varphi}{\partial_z \varphi} [Z^m, \partial_z] f \right\| \lesssim \left\| \frac{\partial_i \varphi}{\partial_z \varphi} \right\|_{L^\infty} \left\| \sum_{|\beta| \leq m-1} c_\beta \partial_z (Z^m f) \right\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\partial_z f\|_{Y^{\frac{m}{2},0}} + |h|_{1,\infty} \right) \|\partial_z f\|_{X^{m-1,0}}.$$

By summing these three terms, we get the results. For $i = 3$,

$$(3.7) \quad Z^m \left(\frac{\partial_z f}{\partial_z \varphi} \right) = \left[Z^m, \frac{1}{\partial_z \varphi}, \partial_z f \right] + \left(Z^m \frac{1}{\partial_z \varphi} \right) \partial_z f + \frac{1}{\partial_z \varphi} (Z^m \partial_z f)$$

$$(3.8) \quad = \partial_3^\varphi(Z^m) f + \left[Z^m, \frac{1}{\partial_z \varphi}, \partial_z f \right] + \left(Z^m \frac{1}{\partial_z \varphi} \right) \partial_z f + \frac{1}{\partial_z \varphi} [Z^m, \partial_z] f.$$

We just replace $\partial_i \varphi$ as 1, so the control is same. \square

3.2. Interior Equations. Applying Z^m to our system, and using commutator estimates, we get following result

3.2.1. *Pressure.*

$$(3.9) \quad Z^m(\nabla^\varphi q) = \nabla^\varphi(Z^m q) - (C_1^m(q), C_2^m(q), C_3^m(q)) = \nabla^\varphi(Z^m q) - C^m(q).$$

Then, by above proposition, we get

$$(3.10) \quad \|C^m(q)\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla q\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|\nabla q\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right).$$

3.2.2. *Divergence-free.*

$$(3.11) \quad Z^m(\nabla^\varphi \cdot v) = \nabla^\varphi \cdot (Z^m v) - \sum_{i=1}^3 C_i^m(v) = \nabla^\varphi \cdot (Z^m v) - C^m(d)$$

and easily,

$$(3.12) \quad \|C^m(d)\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right).$$

3.2.3. *Transportation.* Using divergence free condition, we have

$$(3.13) \quad \partial_t^\varphi + (v \cdot \nabla^\varphi) = \partial_t + (v_y \cdot \nabla_y) + \frac{1}{\partial_z \varphi} (v \cdot N^\varphi - \partial_t \varphi) \partial_z.$$

Applying Z^m ,

$$(3.14) \quad Z^m (\partial_t^\varphi + v \cdot \nabla^\varphi) v = (\partial_t^\varphi + v \cdot \nabla^\varphi) (Z^m v) + T^m(v),$$

$$\text{where } V_z = \frac{1}{\partial_z \varphi} (v \cdot N^\varphi - \partial_t \varphi) \quad \text{and} \quad N^\varphi = (-\nabla \varphi, 1),$$

$$(3.15) \quad T^m(v) = \sum_{i=1}^2 \{ \partial_i v \cdot Z^m v_i + [Z^m, v_i, \partial_i v] \} + [Z^m, V_z, \partial_z v] + (Z^m V_z) \cdot \partial_z v + V_z [Z^m, \partial_z] v.$$

Using propositions and lemmas, we have

$$(3.16) \quad \|T^m(v)\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|\nabla v\|_{X^{m-1,0}} + \|v\|_{X^{m,0}} + |h|_{X^{m-1,\frac{1}{2}}} \right).$$

3.2.4. *Diffusion.*

$$(3.17) \quad 2Z^m \nabla^\varphi \cdot (S^\varphi v) = 2\nabla^\varphi \cdot Z^m (S^\varphi v) - D^m (S^\varphi v),$$

$$\text{where } D^m (S^\varphi v)_i = 2C_j^m (S^\varphi v)_{ij}$$

and

$$(3.18) \quad 2Z^m (S^\varphi v) = Z^m (\partial_i^\varphi v_j + \partial_j^\varphi v_i) = 2S^\varphi (Z^m v) + (C_i^m (v_j) + C_j^m (v_i)) = 2S^\varphi (Z^m v) + \Theta^m(v),$$

$$\text{where } \Theta^m(v)_{ij} = C_i^m (v_j) + C_j^m (v_i).$$

So, the estimate of $\Theta^m(v)$ is same for $C^m(v)$,

$$(3.19) \quad \|\Theta^{k,\alpha}(v)\| \lesssim \Lambda \left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right).$$

3.3. Boundary Equations. Especially, $\alpha_3 = 0$ because we are on the boundary. And all norms are on $\partial S = \mathbb{R}^2$

3.3.1. *Kinematic boundary.*

$$(3.20) \quad Z^m (\partial_t h - v^b \cdot \mathbf{N}) = \partial_t (Z^m h) - \{ [Z^m, v^b, \mathbf{N}] + (Z^m v^b) \cdot \mathbf{N} + v^b \cdot (Z^m \mathbf{N}) \} = 0,$$

$$\partial_t (Z^m h) - (Z^m v^b) \cdot \mathbf{N} - v^b \cdot (Z^m \mathbf{N}) - C^m(KB) = 0 \quad \text{where} \quad C^{k,\alpha}(KB) = [Z^{k,\alpha}, v^b, \mathbf{N}].$$

$$(3.21) \quad \|C^m(KB)\| = \|[Z^m, v^b, \mathbf{N}]\| \lesssim \Lambda \left(\|v^b\|_{Y^{\frac{m}{2},0}} + \|\mathbf{N}\|_{Y^{\frac{m}{2},0}} \right) \left(\|v^b\|_{X^{m-1,0}} + \|\mathbf{N}\|_{X^{m-1,0}} \right).$$

Then by trace inequality, we have

$$\|C^m(KB)\| \lesssim \Lambda \left(\|\nabla v\|_{Y^{\frac{m}{2},0}} + |h|_{Y^{\frac{m}{2},1}} \right) \left(\|v\|_{X^{m-1,\frac{1}{2}}} + \|\nabla v\|_{X^{m-1,-\frac{1}{2}}} + \|h\|_{X^{m-1,1}} \right).$$

3.3.2. Continuity of Stress tensor.

Lemma 3.2. *We have the following estimate for the control of $\nabla v(\cdot, 0)$ by v^b .*

$$(3.22) \quad |\nabla v(\cdot, 0)|_{X^{s,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{s}{2},1}} + \|v\|_{Y^{\frac{s}{2},1}}\right) \left(|h|_{X^{s,1}} + |v(\cdot, 0)|_{X^{s,1}}\right).$$

Proof. We divide ∇v as normal part and tangential part. But for $\partial_1 v$ and $\partial_2 v$, result is obvious. So, we only focus on $\partial_z v$. Firstly, from the divergence free condition $\nabla^\varphi \cdot v = 0$,

$$(3.23) \quad \partial_z v \cdot \mathbf{n} = \frac{1}{|\mathbf{N}|} \left(A + \partial_z \eta \right) \left(\partial_1 v_1 + \partial_2 v_2 \right) \quad \text{where} \quad |\mathbf{N}| = \sqrt{1 + |\nabla \eta|^2}.$$

On the boundary,

$$(3.24) \quad |\partial_z v \cdot \mathbf{n}|_{X^{s,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{s}{2},1}} + \|v\|_{Y^{\frac{s}{2},1}}\right) \left(|\partial_z \eta(\cdot, 0)|_{X^{s,0}} + |v(\cdot, 0)|_{X^{s,1}} \right) \\ \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{s}{2},1}} + \|v\|_{Y^{\frac{s}{2},1}}\right) \left(|h|_{X^{m,1}} + |v(\cdot, 0)|_{X^{s,1}} \right),$$

using estimate of η and trace inequality.

Now, let define $\Pi = I - \mathbf{n} \otimes \mathbf{n}$ (tangential part of vector). We have the following compatibility condition,

$$(3.25) \quad \Pi(S^\varphi v \mathbf{N}) = 0$$

on the boundary. So,

$$(3.26) \quad 2S^\varphi v \mathbf{N} = \frac{1}{\partial_z \varphi} \left(1 + |\nabla h|^2 \right) (\partial_z v) - \left(\partial_1 h \partial_1 v + \partial_2 h \partial_2 v \right) + \begin{pmatrix} \partial_1 v \cdot \mathbf{N} \\ \partial_2 v \cdot \mathbf{N} \\ 0 \end{pmatrix} \mathbf{N} + \frac{1}{\partial_z \varphi} \left(\partial_z v \cdot \mathbf{N} \right) \mathbf{N} = 0,$$

$$(3.27) \quad \partial_z v(\cdot, 0) = \frac{\partial_z \varphi}{1 + |\nabla h|^2} \left\{ \left(\partial_1 h \partial_1 v + \partial_2 h \partial_2 v \right) - \begin{pmatrix} \partial_1 v \cdot \mathbf{N} \\ \partial_2 v \cdot \mathbf{N} \\ 0 \end{pmatrix} \mathbf{N} - \frac{1}{\partial_z \varphi} \left(\partial_z v \cdot \mathbf{N} \right) \mathbf{N} \right\}.$$

We take Π , $|\cdot|_{X^{s,0}}$ and use above $|\partial_z v \cdot \mathbf{n}|_{X^{s,0}}$ estimate again. So we get the same estimate. By adding normal part and tangential part, we finish the lemma. \square

Now we return to the Stress-continuity condition

$$(3.28) \quad Z^m \left\{ \left(q^b - gh + \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) N \right\} - 2\varepsilon Z^m ((S^\varphi v)^b N) = 0.$$

So,

$$(3.29) \quad \left\{ Z^m q^b - g Z^m h - 2\varepsilon \left(S^\varphi(Z^m v) \right)^b - \varepsilon \left(\Theta^m(v) \right)^b \right\} N + \{ q^b - gh - 2\varepsilon (S^\varphi v)^b \} Z^m \mathbf{N} \\ + \left(\nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) Z^m \mathbf{N} + \left(\nabla \cdot Z^m \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \mathbf{N} + C^m(B) = 0,$$

where

$$(3.30) \quad C^m(B) = -C^m(B)_1 + C^m(B)_2 \\ = -2\varepsilon [Z^m, (S^\varphi v)^b, \mathbf{N}] + \left[Z^m, q^b - gh + \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}, \mathbf{N} \right].$$

With estimates

$$(3.31) \quad \|C^m(B)_1\| = 2\varepsilon \|[Z^m, (S^\varphi v)^b, N]\| \\ \lesssim 2\varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} + \|\nabla v\|_{Y^{\frac{m}{2},0}}\right) \left(|h|_{X^{m-1,1}} + |\nabla v(\cdot, 0)|_{X^{m-1,0}} \right) \\ \lesssim 2\varepsilon \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} + \|\nabla v\|_{Y^{\frac{m}{2},0}}\right) \left(|h|_{X^{m-1,1}} + |v^b|_{X^{m-1,1}} \right),$$

by Lemma 3.2.

$$(3.32) \quad \|C^m(B)_2\| = 2\varepsilon \|[Z^m, (S^\varphi v)^b \mathbf{n} \cdot \mathbf{n}, \mathbf{N}]\|.$$

Similarly as $C^m(B)_1$, we get the same estimate. We also estimate $C^m(S)$, where

$$(3.33) \quad Z^m \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} = \frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h < \nabla h, \nabla Z^m h >}{\sqrt{1+|\nabla h|^2}^3} + C^m(S),$$

which is consist of low order polynomials in terms of h . Take a term in this $C^m(S)$, then we take L^2 norm for the highest order, and L^∞ to others. For large $m > 2$, L^∞ can be controlled by the highest order terms by Sobolev embedding. We get

$$(3.34) \quad \|C^m(S)\| \lesssim \Lambda_{m,\infty}(h, v) |h|_{X^{m-1,1}}.$$

4. PRESSURE ESTIMATES

We linearly divide $q = q^E + q^{NS} + q^S$, where q^E solves

$$(4.1) \quad -\Delta^\varphi q^E = \nabla \cdot ((v \cdot \nabla^\varphi) v),$$

$$q^E|_{z=0} = gh.$$

q^{NS} solves

$$(4.2) \quad -\Delta^\varphi q^{NS} = 0,$$

$$q^{NS}|_{z=0} = 2\varepsilon(S^\varphi v) \mathbf{n} \cdot \mathbf{n}.$$

q^S solves

$$(4.3) \quad -\Delta^\varphi q^S = 0,$$

$$q^S|_{z=0} = -\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}.$$

These equations can be transformed into elliptic equation. Gradient becomes,

$$(4.4) \quad \nabla^\varphi f = \begin{pmatrix} \partial_1 f - \frac{\partial_1 \varphi}{\partial_z \varphi} \partial_z f \\ \partial_2 f - \frac{\partial_2 \varphi}{\partial_z \varphi} \partial_z f \\ \frac{\partial_z f}{\partial_z \varphi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\frac{\partial_1 \varphi}{\partial_z \varphi} \\ 0 & 1 & -\frac{\partial_2 \varphi}{\partial_z \varphi} \\ 0 & 0 & \frac{\partial_z f}{\partial_z \varphi} \end{pmatrix} \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_z f \end{pmatrix} = \frac{1}{\partial_z \varphi} P^* \nabla f,$$

where

$$(4.5) \quad P = \begin{pmatrix} \partial_z \varphi & 0 & 0 \\ 0 & \partial_z \varphi & 0 \\ -\partial_1 \varphi & -\partial_2 \varphi & 1 \end{pmatrix}.$$

Similarly, divergence becomes, (easy to check.)

$$(4.6) \quad \nabla^\varphi \cdot v = \frac{1}{\partial_z \varphi} \nabla \cdot (Pv).$$

So, we get easily,

$$(4.7) \quad \Delta^\varphi f = \nabla^\varphi \cdot (\nabla^\varphi f) = \frac{1}{\partial_z \varphi} \nabla \cdot (P \nabla^\varphi f) = \frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla f),$$

where

$$(4.8) \quad E = \frac{1}{\partial_z \varphi} P P^* = \begin{pmatrix} \partial_z \varphi & 0 & -\partial_1 \varphi \\ 0 & \partial_z \varphi & -\partial_2 \varphi \\ -\partial_1 \varphi & -\partial_2 \varphi & \frac{1+(\partial_1 \varphi)^2+(\partial_2 \varphi)^2}{\partial_z \varphi} \end{pmatrix}.$$

We start with two lemmas about Elliptic-Dirichlet boundary problem. These are very similar to those of [1], with some slight modification for our functionspace. First is nonhomogeneous problem with homogeneous boundary data.

Lemma 4.1. *For the system in S ,*

$$(4.9) \quad -\nabla \cdot (E\nabla\rho) = \nabla \cdot F. \quad \rho(t, y, 0) = 0.$$

Then we have the estimates.

$$(4.10) \quad \|\rho\|_{X^{k,0}} + \|\nabla\rho\|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \|\nabla \cdot F\|_{H_{tan}^1} + \|F\|_{H_{tan}^2}\right) \left(\|F\|_{X^{k,0}} + |h|_{X^{k-1,1}}\right) \\ \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \|F\|_{H^2}\right) \left(\|F\|_{X^{k,0}} + |h|_{X^{k-1,1}}\right).$$

Proof. First, we know the basic result,

$$(4.11) \quad \|\nabla\rho\| \leq \Lambda_0 \|F\|.$$

We apply Z^α to the equation, then divergence structure is broken, since Z_3 and ∂_z do not commute. So we apply \tilde{Z}_3 , so that

$$(4.12) \quad \tilde{Z}_3 f = Z_3 f + \frac{1}{(1-z)^2} f$$

then we have,

$$\tilde{Z}_3 \partial_z = \partial_z Z_3.$$

Now we apply $\tilde{Z}^\alpha = Z_t^{\alpha_t} Z_1^{\alpha_1} Z_2^{\alpha_2} \tilde{Z}_3^{\alpha_3}$ to the equation, then

$$(4.13) \quad \nabla \cdot (Z^\alpha (E\nabla\rho)) = \nabla \cdot (Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha)F_h - (\tilde{Z}^\alpha - Z^\alpha)(E\nabla\rho)_h),$$

where $F_h = (F_1, F_2, 0)$. And again,

$$(4.14) \quad \nabla \cdot (E \cdot \nabla (Z^\alpha \rho)) = \nabla \cdot (Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha)F_h - (\tilde{Z}^\alpha - Z^\alpha)(E\nabla\rho)_h) + \nabla \cdot C,$$

where

$$C = -\left(E[Z^\alpha, \nabla]\rho\right) - \left(\sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta,\gamma} Z^\beta E \cdot Z^\gamma \rho\right).$$

Since $Z^\alpha \rho$ is also zero on the boundary,

$$(4.15) \quad \|\nabla\rho\|_{X^{k,0}} \leq \Lambda_0 \left(\|F\|_{X^{k,0}} + \|E\nabla\rho\|_{X^{k-1,0}} + \|E[Z^\alpha, \nabla]\rho\| + \left\| \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta,\gamma} Z^\beta E \cdot Z^\gamma \rho \right\| \right).$$

3 terms on the right hand side can be estimated as follow.

$$(4.16) \quad \|E\nabla\rho\|_{X^{k-1,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},0}} + \|\nabla\rho\|_{Y^{\frac{k}{2},0}}\right) \left(\|\nabla\rho\|_{X^{k-1,0}} + |h|_{X^{k-1,1}}\right)$$

and, since $[Z^\alpha, \partial_z] = \sum_{|\beta| \leq |\alpha|-1} c_{\alpha,\beta} \partial_z (Z^\beta \cdot)$,

$$(4.17) \quad \left\| E \sum_{|\beta| \leq |\alpha|-1} c_{\alpha,\beta} \partial_z (Z^\beta \rho) \right\| \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},0}}\right) \|\nabla\rho\|_{X^{k-1,0}},$$

$$(4.18) \quad \left\| \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta,\gamma} Z^\beta E \cdot Z^\gamma \rho \right\| \lesssim \Lambda\left(|h|_{Y^{\frac{k}{2},1}} + \|\nabla\rho\|_{Y^{\frac{k}{2},0}}\right) \left(\|\nabla\rho\|_{X^{k-1,0}} + |h|_{X^{k-1,1}}\right).$$

Using these 3 estimates we get,

$$(4.19) \quad \|\nabla\rho\|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + \|\nabla\rho\|_{Y^{\frac{k}{2},0}}\right) \left(\|F\|_{X^{k,0}} + \|\nabla\rho\|_{X^{k-1,0}} + |h|_{X^{k-1,1}}\right).$$

Now, we can use induction for $\|\nabla\rho\|_{X^{k-1,0}}$ until $\|\nabla\rho\|$, and $\|\nabla\rho\|_{L^\infty}$ can be estimated as in the [1], (6.21)

$$(4.20) \quad \|\nabla\rho\|_{L^\infty} \leq \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3\right) \left(\|\nabla \cdot F\|_{H_{tan}^1} + \|F\|_{H_{tan}^2}\right).$$

To control $\|\rho\|_{X^{k,0}}$, we smoothly extend ρ to the whole space beyond S , so that it decays to zero at infinity and satisfies $\|\nabla \rho_{ext}\|_{X^{k,0}(\mathbb{R}^3)} \leq C\|\nabla \rho\|_{X^{k,0}(S)}$. Then by Gagliardo-Nirenberg-Sobolev inequality in whole space \mathbb{R}^3 , we get the similar control for $\|\rho\|_{X^{k,0}(S)}$. \square

Indeed, estimate for standard Sobolev space is also available, but since F contains v , F can be estimated in co-normal space. This is why we made estimate in co-normal spaces.

Second is homogeneous problem with non-homogeneous boundary data.

Lemma 4.2. *For the system in S ,*

$$(4.21) \quad -\nabla \cdot (E\nabla \rho) = 0. \quad \rho(t, y, 0) = f^b.$$

Then we have the estimates.

$$(4.22) \quad |\rho|_{X^{k,0}} + |\nabla \rho|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |f^b|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + |f^b|_{X^{k,\frac{1}{2}}}\right).$$

Proof. We divide ρ under the form $\rho = \rho^H + \rho^r$, where ρ^H absorb the boundary data, and ρ^r solves

$$(4.23) \quad -\nabla \cdot (E\nabla \rho^r) = \nabla \cdot (E\nabla \rho^H), \quad \rho^r(t, y, 0) = 0.$$

We choose ρ^H as

$$\hat{\rho}^H(\xi, z) = \chi(z\xi) \hat{f}^b.$$

Then using proposition in section 2 (harmonic extension), we get

$$(4.24) \quad |\nabla \rho^H|_{X^{k,0}} \lesssim C_s |f^b|_{X^{k,\frac{1}{2}}} \\ |\rho^H|_{Y^{k,0}} \lesssim C_s |f^b|_{Y^{k,0}}.$$

Because we can deduce estimate in standard Sobolev space for above lemma, we have

$$(4.25) \quad |\nabla \rho^r|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \left\| \nabla \cdot (E\nabla \rho^H) \right\|_{H_{tan}^1} + \|E\nabla \rho^H\|_{H_{tan}^2}\right) \left(|E\nabla \rho^H|_{X^{k,0}} + |h|_{X^{k-1,1}}\right) \\ \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \|E\nabla \rho^H\|_{H^2}\right) \left(|E\nabla \rho^H|_{X^{k,0}} + |h|_{X^{k-1,1}}\right),$$

where on the right term,

$$(4.26) \quad |E\nabla \rho^H|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},0}} + |f^b|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + |f^b|_{X^{k,\frac{1}{2}}}\right).$$

Consequently, these implies

$$(4.27) \quad |\nabla \rho^r|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |f^b|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + |f^b|_{X^{k,\frac{1}{2}}}\right).$$

We obtained estimates of $|\nabla \rho^r|_{X^{k,0}}$ and $|\nabla \rho^H|_{X^{k,0}}$. Estimates for $|\rho^r|_{X^{k,0}}$ and $|\rho^H|_{X^{k,0}}$ are nearly similar as we explained in Lemma 4.1. \square

These two lemmas give estimate of q^{NS} and q^S .

Proposition 4.3. *Estimates of q^{NS} .*

$$(4.28) \quad |q^{NS}|_{X^{k,0}} + |\nabla q^{NS}|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |\nabla v|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + \varepsilon |h|_{X^{k,\frac{3}{2}}} + \varepsilon \|\nabla v\|_{X^{k,1}} + \varepsilon \|v\|_{X^{k,2}}\right).$$

Proof. Applying above lemma 4.2, $f^b = 2\varepsilon (S^\varphi v)^b \mathbf{n} \cdot \mathbf{n}$,

$$(4.29) \quad |\nabla q^{NS}|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \left| \varepsilon (S^\varphi v)^b \mathbf{n} \cdot \mathbf{n} \right|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + \left| \varepsilon (S^\varphi v)^b \mathbf{n} \cdot \mathbf{n} \right|_{X^{k,\frac{1}{2}}}\right) \\ \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |\nabla v|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + \varepsilon |h|_{X^{k,\frac{3}{2}}} + \varepsilon \|\nabla v(\cdot, 0)\|_{X^{k,\frac{1}{2}}}\right).$$

Using lemma in section3 and trace inequality,

$$\lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |\nabla v|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + \varepsilon |h|_{X^{k,\frac{3}{2}}} + \varepsilon \|\nabla v\|_{X^{k,1}} + \varepsilon \|v\|_{X^{k,2}}\right).$$

□

Proposition 4.4. *Estimate of q^S .*

$$(4.30) \quad |q^S|_{X^{k,0}} + |\nabla q^S|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |h|_{Y^{\frac{k}{2},2}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + |h|_{X^{k,\frac{5}{2}}}\right).$$

Proof. Applying above lemma 4.2, $f^b = -\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}$,

$$(4.31) \quad \begin{aligned} |\nabla q^S|_{X^{k,0}} &\lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + \left|\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + \left|\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}}\right|_{X^{k,\frac{1}{2}}}\right) \\ &\lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{k}{2},1}} + |h|_{2,\infty} + |h|_3 + |h|_{Y^{\frac{k}{2},2}}\right) \left(|h|_{X^{k,\frac{1}{2}}} + |h|_{X^{k,\frac{5}{2}}}\right). \end{aligned}$$

□

To estimate q^E , we do not use above lemma 4.1 directly. This is because, by divergence free condition,

$$\nabla \cdot (v \cdot \nabla v) = \nabla v : (\nabla v)^T,$$

that is, one derivative for v is canceled. This gives 1 more regularity to q^E and this fact make it possible to estimate $\|S_n\|_{X^{m-1,0}}$, in later section. Now we divide $q^E = q_1^E + q_2^E$ as, where q_1^E solves

$$(4.32) \quad -\Delta^\varphi q_1^E = 0, \quad q_1^E(t, y, 0) = gh$$

and q_2^E solves

$$-\Delta^\varphi q_2^E = (\nabla^\varphi v) : (\nabla^\varphi v)^T, \quad q_2^E(t, y, 0) = 0.$$

First, estimate of q_1^E comes from Lemma 4.2 easily.

$$(4.33) \quad \|\nabla q_1^E\|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |h|_{Y^{\frac{k}{2},0}}\right) |h|_{X^{k,\frac{1}{2}}}.$$

Second, for q_2^E , by applying Z^k , we have

$$(4.34) \quad -\nabla \cdot (E \nabla Z^k q_2^E) = \nabla \cdot (E[Z^k, \nabla] q_2^E + [Z^k, E] \nabla q_2^E) + [Z^k, \nabla] \cdot (E \nabla q_2^E) + \sum_{i,j} Z^k (\partial_i^\varphi v_j \partial_j^\varphi v_i).$$

Again, we write $q_2^E = q_{2,1}^E + q_{2,2}^E$ where

$$(4.35) \quad \begin{aligned} -\nabla \cdot (E \nabla Z^k q_{2,1}^E) &= \nabla \cdot (E[Z^k, \nabla] q_{2,1}^E + [Z^k, E] \nabla q_{2,1}^E), \\ -\nabla \cdot (E \nabla Z^k q_{2,2}^E) &= [Z^k, \nabla] \cdot (E \nabla q_{2,2}^E) + \sum_{i,j} Z^k (\partial_i^\varphi v_j \partial_j^\varphi v_i). \end{aligned}$$

$q_{2,1}^E$ can be estimated by Lemma 4.1, and $q_{2,2}^E$ can be estimated by

$$(4.36) \quad \|Z^k q_{2,2}^E\|_{H^2} \lesssim \|[Z^k, \nabla] \cdot (E \nabla q_{2,2}^E) + \sum_{i,j} Z^k (\partial_i^\varphi v_j \partial_j^\varphi v_i)\| + \|Z^k q_{2,2}^E\|_{L^2}.$$

Then by using induction for k and using basic L^2 estimate for $q_{2,2}^E$, (for basic L^2 estimate, we just use (4.11)), we can finish the estimate. Now we put together all estimates of $q_1^E, q_{2,1}^E, q_{2,2}^E$ to get,

Proposition 4.5. *Estimate of q^E*

$$(4.37) \quad \|q^E\|_{X^{k,0}} + \|\nabla q^E\|_{X^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |h|_{Y^{\frac{k}{2}+1,1}} + \|\nabla v\|_{Y^{\frac{k}{2},0}}\right) \left(|h|_{X^{k+1,1}} + \|\nabla v\|_{X^{k-1,0}}\right).$$

We also should estimate L^∞ -type terms of pressure. In fact, for q^{NS} and q^S , we can use Sobolev embedding. For q^E , since we can estimate $\|Z^k q^E\|_{H^2}$, $\|\partial_{zz} q^E\|$ type term can be estimated.

Proposition 4.6. *L^∞ type estimate for pressure.*

$$(4.38) \quad \|\nabla q\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{2,\infty} + |h|_3 + |h|_{Y^{\frac{k}{2}+2,2}} + \|\nabla v\|_{Y^{\frac{k}{2}+1,0}}\right) \left(|h|_{X^{k+3,1}} + \varepsilon \|\nabla v\|_{X^{k+2,1}} + \|\nabla v\|_{X^{k+1,0}}\right).$$

Proof. Using anisotropic Sobolev embedding,

$$(4.39) \quad \|\nabla q^E\|_{Y^{k,0}} \lesssim \|\partial_z \nabla q^E\|_{X^{k+1,0}} \|\nabla q^E\|_{X^{k+2,0}}.$$

Hence, we should estimate $\|\partial_{zz} q^E\|_{X^{k+1,0}}$. Meanwhile, $\|Z^{k+1} q^E\|_{H^2}$ can be estimated by standard theory. For q^{NS} and q^S , standard Sobolev embedding can be used.

$$(4.40) \quad \begin{aligned} \|\nabla q\|_{Y^{k,0}} &\lesssim \|\nabla q^E\|_{Y^{k,0}} + \|\nabla q^{NS}\|_{Y^{k,0}} + \|\nabla q^S\|_{Y^{k,0}} \\ &\lesssim \|\nabla q^E\|_{X^{k+2,0}} \|\partial_z \nabla q^E\|_{X^{k+1,0}} + |\nabla q^{NS}|_{X^{k+2,0}} + |\nabla q^S|_{X^{k+2,0}} \end{aligned}$$

and for $\|\partial_z \nabla q^E\|_{X^{k+1,0}}$, we see that $\|Z^{k+1} q^E\|_{H^2}$ can be estimated by standard theory. Using above lemmas, we can get our result. \square

5. ENERGY ESTIMATES

We perform energy estimate on S . Our terms have forms of

$$(5.1) \quad \int_S f g dV_t \quad \text{where} \quad dV_t = \partial_z \varphi(t, y, z) dy dz.$$

The following lemma is about integration by parts for $\int_S \cdot dV_t$.

Lemma 5.1.

$$(5.2) \quad \int_S \partial_i^\varphi f g dV_t = - \int_S f \partial_i^\varphi g dV_t + \int_{z=0} f g N_i dy, \quad i = 1, 2, 3,$$

$$(5.3) \quad \int_S \partial_t^\varphi f g dV_t = \partial_t \int_S f g dV_t - \int_S f \partial_t^\varphi g dV_t - \int_{z=0} f g \partial_t h,$$

where $\mathbf{N} = (N_1, N_2, N_3)$.

Proof. see [1]. \square

Corollary 5.2. Let $v(t, \cdot)$ is a vector field on S , such that $\nabla^\varphi \cdot v = 0$, then for every smooth f, g and smooth vector field u, w , we have the following estimates.

$$(5.4) \quad \int_S \left(\partial_t^\varphi f + v \cdot \nabla^\varphi f \right) f dV_t = \frac{1}{2} \partial_t \int_S |f|^2 dV_t - \frac{1}{2} \int_{z=0} \left(\partial_t h - v \cdot \mathbf{N} \right) dy,$$

$$(5.5) \quad \int_S \left(\Delta^\varphi f \right) g dV_t = - \int_S \nabla^\varphi f \cdot \nabla^\varphi g dV_t + \int_{z=0} \nabla^\varphi f \cdot \mathbf{N} g dy,$$

$$(5.6) \quad \int_S \nabla^\varphi \cdot \left(S^\varphi u \right) \cdot w dV_t = - \int_S S^\varphi u \cdot S^\varphi w dV_t + \int_{z=0} \left(S^\varphi u \mathbf{N} \right) \cdot w dy.$$

Proof. See [1]. \square

Lemma 5.3. For any smooth solution v, h , we have the basic energy identity.

$$(5.7) \quad \frac{d}{dt} \left(\int_S |v|^2 dV_t + g \int_{z=0} |h|^2 dy + 2 \int_{\partial S} (\sqrt{1 + |\nabla h|^2} - 1) dy \right) + 4\varepsilon \int_S |S^\varphi v|^2 dV_t = 0.$$

Proof. Using above corollary,

$$(5.8) \quad \begin{aligned} \frac{d}{dt} \int_S |v|^2 dV_t &= 2 \int_S \nabla^\varphi \cdot (2\varepsilon S^\varphi v - q) \cdot v dV_t \\ \frac{d}{dt} \int_S |v|^2 dV_t + 4\varepsilon \int_S |S^\varphi v|^2 dV_t &= 2 \int_{\partial S} (2\varepsilon S^\varphi v - qI) \mathbf{N} \cdot v dy \\ &= 2 \int_{\partial S} \left(-gh \mathbf{N} \cdot v + \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \mathbf{N} \cdot v \right) dy \\ &= -g \frac{d}{dt} \int_{\partial S} |h|^2 dy - 2 \int_{\partial S} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \nabla h_t dy \\ &= -g \frac{d}{dt} \int_{\partial S} |h|^2 dy - 2 \frac{d}{dt} \int_{\partial S} (\sqrt{1 + |\nabla h|^2} - 1) dy. \end{aligned}$$

□

As we commented, we work on time interval $\partial_z \varphi$ should be positive and $|h|_{2,\infty}$ should be bounded. So we do calculate energy estimate on an interval of time $[0, T^\varepsilon]$ for which we assume

$$(5.9) \quad \partial_z \varphi \geq c_0, \quad |h|_{2,\infty} \leq \frac{1}{c_0}, \quad \forall t \in [0, T^\varepsilon].$$

Proposition 5.4. *For $t \in [0, T^\varepsilon]$, let us define*

$$(5.10) \quad \Lambda_{m,\infty}(h, f, g, \dots) \doteq \sup_t \Lambda\left(\frac{1}{c_0}, |h|_{Y^{\frac{m}{2},1}} + \|\nabla f\|_{Y^{\frac{m}{2},0}} + \|\nabla g\|_{Y^{\frac{m}{2},0}} + \dots\right),$$

then for every $m \in \mathbb{Z}$, every smooth solutions satisfy the following.

$$(5.11) \quad \|v(t)\|_{X^{m-1,1}}^2 + |h(t)|_{X^{m-1,2}}^2 + \varepsilon \int_0^t \|\nabla v(s)\|_{X^{m-1,1}}^2 ds \leq \Lambda\left(\frac{1}{c_0}, \|v(0)\|_{X^{m-1,1}}, |h(0)|_{X^{m-1,1}}\right) \\ + \int_0^t \Lambda_{m,\infty}(h, v, q) \left(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-2,1}}^2 + |h|_{X^{m-1,\frac{5}{2}}}^2 \right) ds.$$

Proof. Let $Z^m \neq \partial_t^m$. Use above lemma and new equations in section 3. Then we get,

$$(5.12) \quad \frac{d}{dt} \int_S |Z^m v|^2 dV_t + 4\varepsilon \int_S |S^\varphi(Z^m v)|^2 dV_t = 2 \int_{z=0} (2\varepsilon S^\varphi(Z^m v) - (Z^m q)I) \mathbf{N} \cdot (Z^m v) dy + R_S + R_C,$$

where

$$(5.13) \quad R_S = 2\varepsilon \int_S \{ \nabla^\varphi \cdot \Theta^m(v) - D^m(S^\varphi v) \} \cdot Z^m v dV_t,$$

$$(5.14) \quad R_C = 2 \int_S \{ C^m(q) - C^m(T) \} (Z^m v) + C^m(d)(Z^m q) dV_t.$$

And we use the continuity of stress tensor condition for the integral term in the right hand side. From boudnary equation, we have

$$(5.15) \quad 2 \int_{z=0} (2\varepsilon S^\varphi(Z^m v) - (Z^m q)I) \mathbf{N} \cdot (Z^m v) dy \\ = 2 \int_{z=0} \left\{ -g Z^m h + \left(\nabla \cdot \left(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h \cdot \nabla h, \nabla Z^m h}{\sqrt{1+|\nabla h|^2}^3} + C^m(S) \right) \right) \right\} \mathbf{N} \cdot (Z^m v) dy \\ + 2 \int_{z=0} \left\{ (q - gh)I - 2\varepsilon(S^\varphi v) + \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} I \right\} (Z^m \mathbf{N}) \cdot (Z^m v) dy + R_B,$$

where

$$(5.16) \quad R_B = 2 \int_{z=0} (C^m(B) - \varepsilon (\Theta^m(v))^b \mathbf{N}) \cdot (Z^m v) dy.$$

The highest order of h part is

$$(5.17) \quad 2 \int_{z=0} \nabla \cdot \left(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h \cdot \nabla h, \nabla Z^m h}{\sqrt{1+|\nabla h|^2}^3} + C^m(S) \right) (\mathbf{N} \cdot Z^m v) dy.$$

We use Kinematic Boundary condition on the $(\mathbf{N} \cdot Z^m v)$ then focus on

$$-2 \int_{\partial S} \frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} \cdot \nabla (\partial_t(Z^m h)),$$

because this gives

$$(5.18) \quad 2 \int_{\partial S} \frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} \cdot \nabla (\partial_t(Z^m h)) = \frac{d}{dt} \int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}} + \int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}^3} \langle \nabla h, \nabla \partial_t h \rangle.$$

So, whole second part becomes,

$$(5.19) \quad - \left\{ \frac{d}{dt} \int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}} + \int_{\partial S} \frac{|\nabla Z^m h|^2}{\sqrt{1+|\nabla h|^2}^3} \langle \nabla h, \nabla \partial_t h \rangle \right\} + P_1,$$

where P_1 will be given below. We integrate in time, under assuming $|h|_{1,\infty}$ is bounded, then we get,

$$(5.20) \quad \begin{aligned} & \|Z^m v(t)\|_{L^2(S)}^2 + \int_{\partial S} \frac{|Z^m \nabla h(t)|^2}{\sqrt{1+|\nabla h|^2}} dy + g |Z^m h(t)|_{L^2(\partial S)}^2 + 4\epsilon \int_0^t \|S^\varphi(Z^m v)(s)\|_{L^2(S)}^2 ds \\ & \leq \Lambda \left(\frac{1}{c_0}, \|(Z^m v)(0)\|_{L^2(S)}, |(Z^m \nabla h)(0)|_{L^2(\partial S)}, g |Z^m h(0)|_{L^2(\partial S)}^2 \right) + \Lambda_0 \int_0^t |(R_S + R_C + R_B + P_1 + P_2 + P_3)| ds \\ & \quad + \Lambda_0 \int_0^t \int_{\partial S} |\nabla Z^m h|^2 |\langle \nabla h, \nabla \partial_t h \rangle| dA ds, \end{aligned}$$

where R_S, R_C, R_B were defined in (5.13), (5.14) and (5.16). And

$$(5.21) \quad \begin{aligned} P_1 & \doteq 2 \int_{\partial S} \left(\frac{\nabla Z^m h}{\sqrt{1+|\nabla h|^2}} - \frac{\nabla h \langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1+|\nabla h|^2}^3} + C^m(S) \right) \cdot \nabla \left(v^b \cdot (Z^m \mathbf{N}) + C^m(KB) \right) dy \\ & \quad + 2 \int_{\partial S} \left(\frac{\nabla h \langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1+|\nabla h|^2}^3} - C^m(S) \right) \cdot \nabla \left(\partial_t(Z^m h) \right) dy, \end{aligned}$$

$$(5.22) \quad P_2 \doteq 2 \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varphi v) + \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right\} (Z^m \mathbf{N}) \cdot (Z^m v) dy,$$

$$(5.23) \quad P_3 \doteq 2g \int_{z=0} Z^m h \left(v^b \cdot Z^m \mathbf{N} + C^m(KB) \right) dy.$$

Now we should estimate above six terms.

1) Estimate of R_B

$$(5.24) \quad \begin{aligned} |R_B| &= 2 \left| \int_{z=0} C^m(B) - \varepsilon \left(\Theta^m(v) \right)^b (v) \mathbf{N} \cdot (Z^m v) dy \right| \\ &\lesssim \left\| C^m(B) - \varepsilon \left(\Theta^m(v) \right)^b \mathbf{N} \right\|_{L^2(\partial S)} \|Z^m v\|_{L^2(\partial S)} \\ &\lesssim \varepsilon \Lambda_{m,\infty}(h, v) \left(|h|_{X^{m-1,1}} + |v^b|_{X^{m,\frac{1}{2}}} \right) \|v\|_{X^m}. \end{aligned}$$

2) Estimate of P_2

$$(5.25) \quad \begin{aligned} P_2 &\doteq 2 \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varphi v) + \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right\} (Z^m \mathbf{N}) \cdot (Z^m v) dy \\ &= 2 \int_{z=0} \{ q^{NS}|_{z=0} I - 2\varepsilon(S^\varphi v) \} (Z^m \mathbf{N}) \cdot (Z^m v) dy = 4\varepsilon \int_{z=0} \{ (S^\varphi v) \mathbf{n} \cdot \mathbf{n} I - (S^\varphi v) \} (Z^m \mathbf{N}) \cdot (Z^m v) dy. \end{aligned}$$

So,

$$(5.26) \quad \begin{aligned} |P_2| &= \left| 2 \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varphi v) + \nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right\} (Z^m \mathbf{N}) \cdot (Z^m v) dy \right| \\ &\lesssim 2\varepsilon |Z^m \mathbf{N}|_{-\frac{1}{2}} |\{ (S^\varphi v) \mathbf{n} \cdot \mathbf{n} I - (S^\varphi v) \} (Z^m v)|_{\frac{1}{2}} \\ &\lesssim \varepsilon |h|_{X^{m,\frac{1}{2}}} |v^b|_{X^{m,\frac{1}{2}}} |(S^\varphi v) \mathbf{n} \cdot \mathbf{n} I - (S^\varphi v)|_{1,\infty} \\ &\lesssim \varepsilon \Lambda_{m,\infty}(h, v) |h|_{X^{m,\frac{1}{2}}} |v^b|_{X^{m,\frac{1}{2}}}. \end{aligned}$$

3) Estimate of P_3

$$(5.27) \quad P_3 \doteq 2g \int_{z=0} Z^m h \left(v^b \cdot Z^m \mathbf{N} + C^m(KB) \right) dy,$$

$$(5.28) \quad \begin{aligned} |P_3| &\leq \left| \int_{z=0} Z^m h \left(v^b \cdot Z^m \mathbf{N} + C^m(KB) \right) dy \right| \leq \left| Z^m h \cdot \left(v^b \cdot Z^m \mathbf{N} + C^m(KB) \right) \right|_{L^1(\partial S)} \\ &\leq |Z^m h(t)|_{L^2(\partial S)} \left\{ |v^b \cdot Z^m \mathbf{N}|_{L^2(\partial S)} + |C^m(KB)(t)|_{L^2(\partial S)} \right\} \\ &\lesssim |h|_{X^{m,0}} \left\{ |v^b|_{L^\infty} |h|_{X^{m,1}} + \|C^m(KB)\| \right\} \\ &\lesssim \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,1}}^2 \right). \end{aligned}$$

4) Estimate of R_C

$$(5.29) \quad R_C = 2 \int_S \{C^m(q) - T^m(v)\} (Z^m v) + C^m(d) (Z^m q) dV_t,$$

$$(5.30) \quad \begin{aligned} |R_C| &\lesssim \Lambda_0 \left(\|C^m(d)\|_{L^2} \|Z^m q(t)\|_{L^2} + \|T^m(v)\|_{L^2} \|Z^m v(t)\|_{L^2} + \|C^m(q)\|_{L^2} \|Z^m v(t)\|_{L^2} \right) \\ &\lesssim \Lambda_0 \left(\|C^m(d)\| \|q(t)\|_{X^{m,0}} + \|T^m(v)\| \|v(t)\|_{X^{m,0}} + \|C^m(q)\| \|v(t)\|_{X^{m,0}} \right) \\ &\lesssim \Lambda_{m,\infty}(h, v, q) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,\frac{1}{2}}}^2 + \|q\|_{X^{m,0}}^2 + \|\nabla q\|_{X^{m-1,0}}^2 \right). \end{aligned}$$

5) Estimate of R_S

$$(5.31) \quad \begin{aligned} R_S &= 2\varepsilon \int_S \{ \nabla^\varphi \cdot \Theta^m(v) - D^m(S^\varphi v) \} \cdot Z^m v dV_t \\ &= -2\varepsilon \int_S \Theta^m(v) : \nabla^\varphi (Z^m v) + 2\varepsilon \int_S \Theta^m(v) N \cdot (Z^m v^b) + 2\varepsilon \sum_{i,j} \int_S C_j^m(S^\varphi v)_{ij} (Z^m v)_i dV_t. \end{aligned}$$

We have only two types of integrals. ($m = m_1 + m_2$ and both are non-zero indices.)

$$(5.32) \quad I_1 \doteq \int_S \partial_z (Z^m v_i) Z^{m_1} (S^\varphi v)_{ij} Z^{m_2} \left(\frac{\partial_i \varphi}{\partial_z \varphi} \right),$$

$$(5.33) \quad I_2 \doteq \int_{\partial S} (Z^m v_i) Z^{m_1} (S^\varphi v)_{ij} Z^{m_2} \left(\frac{\partial_i \varphi}{\partial_z \varphi} \right).$$

For I_1 , we give L^2 estimate to $\partial_z (Z^m v_i)$, L^2 to bigger m_i , and L^∞ to smaller m_j . So we get

$$(5.34) \quad |I_1| \lesssim \Lambda_{m,\infty} \left(\|\nabla v\|_{X^{m-1,0}}^2 + \|S^\varphi v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}}^2 \right),$$

$$(5.35) \quad |I_2| \lesssim \Lambda_{m,\infty} \left(\|v\|_{X^m}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}}^2 \right).$$

Hence, we have

$$(5.36) \quad |R_S| \lesssim \varepsilon \Lambda_{m,\infty} \left(\|v\|_{X^m}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + \|S^\varphi v\|_{X^{m-1,0}}^2 + |h|_{X^{m-1,\frac{1}{2}}}^2 \right).$$

6) Estimate of P_1

Let $P_1 = P_{1,1} + P_{1,2}$ where

$$(5.37) \quad P_{1,1} \doteq 2 \int_{\partial S} \left(\frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla h, \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}^3} + C^m(S) \right) \cdot \nabla \left(v^b \cdot (Z^m \mathbf{N}) + C^m(KB) \right) dy$$

and

$$P_{1,2} = 2 \int_{\partial S} \left(\frac{\nabla h \cdot \nabla h, \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}^3} - C^m(S) \right) \cdot \nabla \left(\partial_t (Z^m h) \right) dy.$$

Let us treat $P_{1,2} \doteq P_{1,2,1} + P_{1,2,2}$ where

$$\begin{aligned}
 (5.38) \quad P_{1,2,1} &\doteq 2 \int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1 + |\nabla h|^2}^3} \langle \nabla h, \nabla Z^m \partial_t h \rangle dy = P_{1,2,1,1} + P_{1,2,1,2} + P_{1,2,1,3} \\
 &\doteq \frac{d}{dt} \int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h \rangle^2}{\sqrt{1 + |\nabla h|^2}^3} dy - 2 \int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1 + |\nabla h|^2}^3} \langle \nabla \partial_t h, \nabla Z^m h \rangle dy \\
 &\quad + 3 \int_{\partial S} \frac{\langle \nabla h, \nabla \partial_t h \rangle}{\sqrt{1 + |\nabla h|^2}^3} \langle \nabla h, \nabla Z^m h \rangle^2 dy, \\
 P_{1,2,2} &= -2 \int_{\partial S} C^m(S) \cdot \nabla \partial_t Z^m h dy.
 \end{aligned}$$

$P_{1,2,1,1} = \frac{d}{dt} \int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h \rangle^2}{\sqrt{1 + |\nabla h|^2}^3} dy$ is absorbed to energy of left hand side under the assumption that $|h|_{1,\infty}$ is bounded. This is because, after we send this term to the left hand side, we have

$$\int_{\partial S} \frac{|Z^m \nabla h(t)|^2}{\sqrt{1 + |\nabla h|^2}^3} dy \leq \int_{\partial S} \frac{|Z^m \nabla h(t)|^2}{\sqrt{1 + |\nabla h|^2}} dy - \int_{\partial S} \frac{\langle \nabla h, \nabla Z^m h \rangle^2}{\sqrt{1 + |\nabla h|^2}^3} dy.$$

Control of $P_{1,2,1,2}$ and $P_{1,2,1,3}$ are trivial as following,

$$(5.39) \quad |P_{1,2,1,2} + P_{1,2,1,3}| \leq \Lambda_{m,\infty}(h, v) |h|_{X^{m,1}}^2.$$

To estimate $P_{1,2,2}$, we use integration by part in space to get,

$$\begin{aligned}
 (5.40) \quad |P_{1,2,2}| &\leq \left| \int_{\partial S} C^m(S) \cdot \nabla (\partial_t Z^m h) dy \right| = \left| \int_{\partial S} \nabla \cdot C^m(S) \partial_t Z^m h dy \right| \\
 &\leq \|\nabla \cdot C^m(S)\| |\partial_t h|_{X^{m,0}} \leq \Lambda_{m,\infty}(h, v) |h|_{X^{m,1}} (\|v\|_{X^{m,0}} + |h|_{X^{m,1}}).
 \end{aligned}$$

For $P_{1,1}$, it can be controlled by

$$\begin{aligned}
 (5.41) \quad |P_{1,1}| &\leq \left| \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \langle \nabla h, \nabla Z^m h \rangle}{\sqrt{1 + |\nabla h|^2}^3} + C^m(S) \right|_{\frac{1}{2}} |v^b \cdot (Z^m N) + C^m(KB)|_{\frac{1}{2}} \\
 &\leq \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,\frac{3}{2}}}^2 \right).
 \end{aligned}$$

Hence, by putting together all the estimates, we get the estimate of P_1 (after $P_{1,2,1,1}$ is absorbed to energy of left hand side).

$$(5.42) \quad |P_1| \leq \Lambda_{m,\infty}(h, v) \Lambda\left(\frac{1}{c_0}, |h|_{X^{m,1}}\right) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,\frac{3}{2}}}^2 \right).$$

7) Estimate of $\Lambda_0 \int_0^t \int_{\partial S} |\nabla Z^m h|^2 \langle \nabla h, \nabla \partial_t h \rangle dA ds$.

$$(5.43) \quad \int_{\partial S} |\nabla Z^m h|^2 |\langle \nabla h, \nabla \partial_t h \rangle| dA \leq \left| \left\langle \nabla h, \nabla (v^b \cdot \mathbf{N}) \right\rangle \right|_{L^\infty} |h|_{X^{m,1}}^2 \leq \Lambda_{m,\infty}(h, v) |h|_{X^{m,1}}^2.$$

Now, we can gather all estimates. We do not include $Z^m = \partial_t^m$ case. Hence, we consider only the case that at least one of Z^m is spatial derivatives. So our function space is $X^{m-1,1}$. Meanwhile, on the left hand side, we can use proposition 2.6, to replace $S^\varphi v$ into ∇v under the assumption of $|h|_{2,\infty}$ is bounded and $\partial_z \varphi$ is positive. We use trace estimate for $|v^b|_{X^{m,\frac{1}{2}}}$, to get

$$\begin{aligned}
 (5.44) \quad &\|Z^m v(t)\|_{L^2(S)}^2 + |Z^m \nabla h(t)|_{L^2(\partial S)}^2 + |Z^m h(t)|_{L^2(\partial S)}^2 + 4\varepsilon \int_0^t \|S^\varphi(Z^m v)(s)\|_{L^2(S)}^2 ds \\
 &\leq \Lambda\left(\frac{1}{c_0}, \|(Z^m v)(0)\|_{L^2(S)}, |(Z^m \nabla h)(0)|_{L^2(\partial S)}^2, g |Z^m h(0)|_{L^2(\partial S)}^2\right) \\
 &\quad + \int_0^t \Lambda_{m,\infty}(h, v, q) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m,\frac{3}{2}}}^2 + \|\nabla q\|_{X^{m-1,0}}^2 \right) ds.
 \end{aligned}$$

We do not include $Z^m = \partial_t^m$ case. Hence, we consider only the case that at least one of Z^m is spatial derivatives. So our function space is $X^{m-1,1}$. Meanwhile, on the left hand side, we can use proposition 2.6, to replace $S^\varphi v$ into ∇v under the assumption of $|h|_{2,\infty}$ is bounded and $\partial_z \varphi$ is positive. We also use pressure estimate on this estimate. Moreover we also use Young's inequality to separate dissipation type term $\varepsilon \int_0^t \|\nabla v\|_{X^{m-1,1}}$, and then make it absorbed into the energy terms of the left hand side. This finishes the proof. \square

In the next section, we estimate for $Z^m = \partial_t^m$ case. By summing with above estimate, we get the estimate for norm $\|\cdot\|_{X^{m,0}}$.

6. ENERGY ESTIMATES OF ALL TIME-DERIVATIVES

In the section, we treat special case, $Z^m = \partial_t^m$. According to the energy estimate of the previous section, we investigate two terms which generate $\frac{1}{2}$ regularity loss. We find special cancellation to show that regularity of h of commutators is not bad in this all time-derivatives case. We start with a simple proposition in the case of $Z^m = \partial_t^m$. We will use critical cancellation idea from [18], and we thanks to Yanjin Wang.

Proposition 6.1. *When $Z^m = \partial_t^m$, $C^m(f)$ can be estimated as follow.*

$$(6.1) \quad \|C^m(f)\| \leq \Lambda_{m,\infty}(h, f) \left(|\partial_t^m h|_{\frac{1}{2}} + \|\partial_t^{m-1} \partial_z f\| \right).$$

Proof. Since, ∂_t commutes with ∂_z , we get the following.

$$(6.2) \quad \begin{aligned} \partial_t^m (\partial_i^\varphi f) &= \partial_i^\varphi (\partial_t^m f) + C_i^m(f), \\ C_i^m(f) &= - \left[\partial_t^m, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f \right] - \left(\partial_t^m \frac{\partial_i \varphi}{\partial_z \varphi} \right) \partial_z f. \end{aligned}$$

And we get easily,

$$(6.3) \quad \begin{aligned} \left\| \left[\partial_t^m, \frac{\partial_i \varphi}{\partial_z \varphi}, \partial_z f \right] \right\| &\leq \left\| \partial_t^{m-1} \frac{\partial_i \varphi}{\partial_z \varphi} \right\| \left\| \partial_t^{\frac{m}{2}} \partial_z f \right\| + \left\| \partial_t^{\frac{m}{2}} \frac{\partial_i \varphi}{\partial_z \varphi} \right\| \left\| \partial_t^{m-1} \partial_z f \right\| \\ &\leq \Lambda_{m,\infty}(h, f) \left(|\partial_t^{m-1} h|_{\frac{1}{2}} + \|\partial_t^{m-1} \partial_z f\| \right), \end{aligned}$$

and

$$(6.4) \quad \left\| \left(\partial_t^m \frac{\partial_i \varphi}{\partial_z \varphi} \right) \partial_z f \right\| \leq |\partial_z f|_{L^\infty} \left\| \partial_t^m \frac{\partial_i \varphi}{\partial_z \varphi} \right\|.$$

Putting together, we get the result. \square

Proposition 6.2. *For $Z^m = \partial_t^m$, then we have the following energy estimate.*

$$(6.5) \quad \begin{aligned} &\|\partial_t^m v(t)\|_{L^2(S)}^2 + |\partial_t^m \nabla h(t)|_{L^2}^2 + 4\varepsilon \int_0^t \|\nabla \partial_t^m v(s)\|_{L^2(S)}^2 ds \\ &\leq \Lambda \left(\frac{1}{c_0}, \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|(\partial_t^{m-1} \nabla h)(0)\|_{\frac{3}{2}} \right) + \Lambda_{m,\infty}(h, v) \|q(t)\|_{X^{m-1,1}} \\ &+ \Lambda_{m,\infty}(h, v) \int_0^t \|v\|_{X^{m,0}} \|q\|_{X^{m-1,1}} + \Lambda_{m,\infty}(h, v) \|q(t)\|_{X^{m-1,1}} \int_0^t \left(\|\partial_t^m v\|_{L^2} + \|\partial_t^{m-1} v\|_{L^2} + \|\partial_t^m \nabla h\|_{L^2} \right) ds. \end{aligned}$$

Remark 6.3. If we do similar estimate as previous section, we will see $|h|_{X^{m,\frac{3}{2}}}$ on the right hand side. This can not be controlled. The main importance of this section is that $|h|_{X^{m,\frac{3}{2}}}$ does not appear on the right hand side.

Proof. From the analysis of previous section, there are two terms which generate highest order terms. **First term** is $-\int_S C^m(d) \partial_t^m q \partial_z \varphi dy dz$ which reflect pressure estimate part, see (5.14). The **Second term** is direct surface tension effect $-\int_{\partial S} \partial_t^m \left(\nabla \cdot \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) C^m(KB)$, see (5.37).

Let us analyze **First term**. Using divergence free condition, we can expand $\partial_z \varphi C^m(d)$ as follow.

$$(6.6) \quad \begin{aligned} \partial_z \varphi C^m(d) &= [\partial_t^m, \mathbf{N}, \cdot \partial_z v] + [\partial_t^m, \partial_z \eta, \partial_1 v_1 + \partial_2 v_2] \\ &= C^m(d)_1 + C^m(d)_2 + C^m(d)_3 + C^m(d)_4 + C^m(d)_5, \end{aligned}$$

where

$$\begin{aligned}
 C^m(d)_1 &\doteq m\partial_t \mathbf{N} \cdot \partial_t^{m-1} \partial_z v, \\
 C^m(d)_2 &\doteq m\partial_t \partial_z \eta \partial_t^{m-1} (\partial_1 v_1 + \partial_2 v_2), \\
 C^m(d)_3 &\doteq m\partial_t^{m-1} \mathbf{N} \cdot \partial_t \partial_z v, \\
 C^m(d)_4 &\doteq m\partial_t^{m-1} \partial_z \eta \partial_t (\partial_1 v_1 + \partial_2 v_2), \\
 C^m(d)_5 &\doteq \sum_{l=2}^{m-2} C_m^l \left(\partial_t^l \mathbf{N} \cdot \partial_t^{m-l} \partial_z v + \partial_t^l \partial_z \eta \cdot \partial_t^{m-l} (\partial_1 v_1 + \partial_2 v_2) \right).
 \end{aligned} \tag{6.7}$$

We attack these terms including time integrals. We use the following equation

$$\begin{aligned}
 - \int_0^t \int_S C^m(d)_i \partial_t^m q dy dz ds &= \int_0^t \int_S \partial_t C^m(d)_i \partial_t^{m-1} q dy dz ds \\
 - \int_S C^m(d)_i(t) \partial_t^{m-1} q(t) dy dz &+ \int_S C^m(d)_i(0) \partial_t^{m-1} q(0) dy dz,
 \end{aligned}$$

for all $i = 2, 3, 4, 5$.

1) Estimate of $-\int_0^t \int_S C^m(d)_2 \partial_t^m q dy dz$.

$$\begin{aligned}
 - \int_0^t \int_S C^m(d)_2 \partial_t^m q dy dz ds &\leq \Lambda\left(\frac{1}{c_0}, \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|q(0)\|_{X^{m-1,1}}\right) \\
 - \int_S C^m(d)_2(t) \partial_t^{m-1} q(t) dy dz &+ \int_0^t \int_S \partial_t C^m(d)_2 \partial_t^{m-1} q dy dz ds.
 \end{aligned} \tag{6.8}$$

1-1) Estimate of $-\int_S C^m(d)_2(t) \partial_t^{m-1} q(t) dy dz$ at some t .

$$\begin{aligned}
 - \int_S C^m(d)_2(t) \partial_t^{m-1} q(t) dy dz &= - \sum_{i=1}^2 \int_S m \partial_t \partial_z \eta \partial_t^{m-1} \partial_i v_i \partial_t^{m-1} q dy dz \\
 &= \sum_{i=1}^2 \int_S m \partial_t^{m-1} v_i \partial_i (\partial_t \partial_z \eta \partial_t^{m-1} q) dy dz \\
 &\leq \Lambda_m(h, v) \|\partial_t^{m-1} v\|_{L^2} \left(\|\partial_t^{m-1} q\|_{L^2} + \|\partial_t^{m-1} \partial_i q\|_{L^2} \right) \\
 &= \Lambda_m(h, v) \left(\|\partial_t^{m-1} q\|_{L^2} + \|\partial_t^{m-1} \partial_i q\|_{L^2} \right) \left(\|v(0)\|_{X^{m-1,0}} + \int_0^t \|\partial_t^m v\|_{L^2} ds \right) \\
 &\leq \Lambda_m(h, v) \left(\|\partial_t^{m-1} q\|_{L^2} + \|\partial_t^{m-1} \partial_i q\|_{L^2} \right) + \Lambda_m(h, v) \left(\|\partial_t^{m-1} q\|_{L^2} + \|\partial_t^{m-1} \partial_i q\|_{L^2} \right) \int_0^t \|\partial_t^m v\|_{L^2} ds.
 \end{aligned}$$

where θ is sufficiently small. In the last step we used Young's inequality.

1-2) Estimate of $\int_0^t \int_S \partial_t C^m(d)_2 \partial_t^{m-1} q dy dz ds$. We perform integration by part in ∂_1 and ∂_2 as like above to get,

$$\begin{aligned}
 \int_0^t \int_S \partial_t C^m(d)_2 \partial_t^{m-1} q dy dz ds &= \sum_{j=1,2} \int_0^t \int_S m \partial_t (m \partial_t \partial_z \eta \partial_t^{m-1} v_j) \partial_t^{m-1} \partial_j q dy dz ds \\
 &\leq \Lambda_{m,\infty}(h, v) \|q(t)\|_{X^{m-1,1}} \int_0^t \left(\|\partial_t^m v\|_{L^2} + \|\partial_t^{m-1} v\|_{L^2} \right).
 \end{aligned}$$

2) Estimate of $-\int_0^t \int_S C^m(d)_3 \partial_t^m q dy dz$. This is similar to previous estimate.

$$\begin{aligned}
 - \int_0^t \int_S C^m(d)_3 \partial_t^m q dy dz ds &\leq \Lambda\left(\frac{1}{c_0}, \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|q(0)\|_{X^{m-1,0}}\right) \\
 - \int_S C^m(d)_3(t) \partial_t^{m-1} q(t) dy dz &+ \int_0^t \int_S \partial_t C^m(d)_3 \partial_t^{m-1} q dy dz ds.
 \end{aligned} \tag{6.9}$$

2-1) Estimate of $-\int_S C^m(d)_3(t)\partial_t^{m-1}q(t)dydz$. We perform similar work as above to get,

$$\int_S C^m(d)_3(t)\partial_t^{m-1}q(t)dydz \leq \Lambda_{m,\infty}(h,v)\|q(t)\|_{X^{m-1,1}} + \Lambda_{m,\infty}(h,v)\|q(t)\|_{X^{m-1,1}} \int_0^t \|\partial_t^m \nabla h\|_{L^2}.$$

2-2) Estimate of $\int_0^t \int_S \partial_t C^m(d)_3 \partial_t^{m-1} q dydz ds$. By regularization effect of η , we do not have to take integration by part. Directly, we get

$$\int_0^t \int_S \partial_t C^m(d)_3 \partial_t^{m-1} q dydz ds \leq \Lambda_{m,\infty}(h,v)\|q(t)\|_{X^{m-1,1}} \int_0^t \|\partial_t^m \nabla h\|_{L^2}.$$

3) Estimate of $-\int_0^t \int_S C^m(d)_4 \partial_t^m q dydz$. This estimate is same as $C^m(d)_3$ case, since η is $\frac{1}{2}$ better than h . We skip the detail.

4) Estimate of $-\int_0^t \int_S C^m(d)_5 \partial_t^m q dydz$. This estimate is even more easier than previous cases and the result is same as before. We are suffice to consider L^2 type norm for v and h both.

4-1) Estimate of $-\int_S C^m(d)_5(t)\partial_t^{m-1}q(t)dydz$.

$$\int_S C^m(d)_5(t)\partial_t^{m-1}q(t)dydz \leq \Lambda_m(h,v)\|\partial_t^{m-1}q\|_{H^1} + \Lambda_m(h,v)\|\partial_t^{m-1}q\|_{H^1} \int_0^t (\|v\|_{X^{m-1,1}} + \|h\|_{X^{m,1}}) ds.$$

4-2) Estimate of $\int_0^t \int_S \partial_t C^m(d)_5 \partial_t^{m-1} q dydz ds$. As before,

$$\int_0^t \int_S \partial_t C^m(d)_5 \partial_t^{m-1} q dydz ds \leq \Lambda_{m,\infty}(h,v)\|\partial_t^{m-1}q\|_{H^1} \int_0^t (\|v\|_{X^{m-1,1}} + \|h\|_{X^{m,1}}) ds.$$

5) Estimate of $-\int_0^t \int_S C^m(d)_1 \partial_t^m q dydz$.

$$-\int_0^t \int_S m \partial_t \mathbf{N} \cdot \partial_t^{m-1} \partial_z v \partial_t^m q.$$

If we intergrate by part in z , we get

$$= -\int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b + \int_0^t \int_S m \partial_z \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^m q + \int_0^t \int_S m \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^m \partial_z q.$$

We use integration by part in time again to get,

$$\begin{aligned} &= -\int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b + \int_0^t \int_S (\text{low order commutators}) \\ &\quad + \left[\int_S m \partial_z \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^{m-1} q \right]_0^t - \int_0^t \int_S m \partial_z \partial_t \mathbf{N} \partial_t^m v \partial_t^{m-1} q \\ &\quad + \left[\int_S m \partial_t \mathbf{N} \partial_t^{m-1} v \partial_t^{m-1} \partial_z q \right]_0^t - \int_0^t \int_S m \partial_t \mathbf{N} \partial_t^m v \partial_t^{m-1} \partial_z q. \end{aligned}$$

For last four terms, we can do similar work as we did in 2) and 3). We do not treat first term since we will use it as cancellation with another highest term. We get,

$$\begin{aligned} &-\int_0^t \int_S C^m(d)_1 \partial_t^m q dydz \leq \Lambda\left(\frac{1}{c_0}, E^{m-1}(0), \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|\partial_t^{m-1} \nabla q(0)\|\right) \\ &\quad - \int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b + \Lambda_{m,\infty}(h,v) \int_0^t \|v\|_{X^{m,0}} \|q\|_{X^{m-1,1}} \\ &\quad + \int_S m (\partial_z \partial_t \mathbf{N} + \partial_t \mathbf{N}) \partial_t^{m-1} v (\partial_t^{m-1} q + \partial_t^{m-1} \partial_z q)(t) dydz. \end{aligned}$$

To treat the last term, note that $\|\partial_t^{m-1} v\| = \|\int_0^t \partial_t^m v(s) ds + \partial_t^{m-1} v(0)\|$ as we did in previous cases. This gives the following.

$$\begin{aligned} &-\int_0^t \int_S C^m(d)_1 \partial_t^m q dydz \leq \Lambda\left(\frac{1}{c_0}, E^{m-1}(0), \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|q(0)\|_{X^{m-1,1}}\right) \\ &\quad - \int_0^t \int_{\partial S} m \partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b + \Lambda_{m,\infty}(h,v) \int_0^t \|v\|_{X^{m,0}} \|q\|_{X^{m-1,1}} \end{aligned}$$

$$+ \Lambda_{m,\infty}(h, v) \|\partial_t^{m-1} q\|_{H^1} \int_0^t \|\partial_t^m v\|_{L^2} ds.$$

Now let us analyze **Second term** on the right hand side. What we should consider is

$$- \int_{\partial S} \partial_t^m \left(\nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) C^m(KB).$$

Highest commutator in $C^m(KB)$ is $m\partial_t N(\partial_t^{m-1} v^b)$. Hence main term we should consider is

$$\begin{aligned} & - \int_{\partial S} \partial_t^m \left(\nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) m\partial_t \mathbf{N}(\partial_t^{m-1} v^b) \\ & = \int_{\partial S} \partial_t^m (q^S)^b m\partial_t \mathbf{N}(\partial_t^{m-1} v^b). \end{aligned}$$

Here we use critical cancellation idea of [18]. This term vanishes highest part of $-\int_0^t \int_{\partial S} m\partial_t \mathbf{N} \partial_t^{m-1} v^b \partial_t^m q^b$ in the estimate of $-\int_0^t \int_S C^m(d)_1 \partial_t^m q^S dydz$. Note that q^E and q^{NS} are not harmful terms. We omit another trivial low order commutator estimates here because we do not have to take any additional integration by parts. Standard Holer inequality is sufficient to close energy estimate. Putting everything previous estimates in this section with pressure estimates of section 3 and Lemma 3.2, we get the result. \square

Meanwhile, if we make $L^\infty L^2$ type estimate in this step, we should be able to control $\|\nabla h\|_{L^\infty X^{m-1,3/2}}$ by $\|\nabla \partial_t h\|_{L^\infty X^{m-1,0}}$. In the next section, however, we will see that what we control is $\|\nabla h\|_{L^2 X^{m-1,3/2}}$, not $L^\infty X^{m-1,3/2}$. This make us to give up $L^\infty L^2$ type energy in this step. At last we have to get energy for all-time derivatives case with L_T^p with $p > 2$. The most easy way to get such one is to square and integrate in time one more to get L_T^4 type energy.

Proposition 6.4. *For $Z^m = \partial_t^m$, then we have the following L^4 in time type energy estimate.*

$$\begin{aligned} (6.10) \quad & \int_0^t \|\partial_t^m v(s)\|_{L^2(S)}^4 ds + (1 + \int_0^t |\partial_t^m \nabla h(s)|_{L^2}^4 ds) + 4\varepsilon \int_0^t \left(\int_0^s \|\nabla \partial_t^m v(\tau)\|_{L^2(S)}^2 d\tau \right)^2 ds \\ & \leq (1+t) \Lambda \left(\frac{1}{c_0}, \|v(0)\|_{X^{m,0}}, |h(0)|_{X^{m,1}}, \|(\partial_t^{m-1} \nabla h)(0)\|_{\frac{3}{2}} \right) + \Lambda_{m,\infty}(h, v) \int_0^t \|q(s)\|_{X^{m-1,1}}^2 ds \\ & \quad + \Lambda_{m,\infty}(h, v) \int_0^t s^{1/2} \|v\|_{L^4 X^{m,0}}^2 \|q\|_{L^2 X^{m-1,1}}^2 ds \\ & \quad + \Lambda_{m,\infty}(h, v) \int_0^t \left(\|q(s)\|_{X^{m-1,1}}^2 \int_0^s \|\partial_t^m v(\tau)\|_{L^2}^2 + \|\partial_t^{m-1} v(\tau)\|_{L^2}^2 + \|\partial_t^m \nabla h(\tau)\|_{L^2}^2 d\tau \right) ds. \end{aligned}$$

Proof. We square proposition 6.2, and integrate in time. We used Jensen's inequaltiy and $L^4 L^4 L^2$ type Holder inequality for $\int_0^t \|v\|_{X^{m,0}} \|q\|_{X^{m-1,1}}$. We add 1 to both sides. \square

7. DIRICHLET-NEUMANN OPERATOR ESTIMATE ON THE BOUNDARY

In this section, we claim that, on the boundary $\partial_x^{3/2} h$ can be controlled by $\partial_t h$ with help of some low order terms, so that we can close the energy estimate. We start with section with a lemma which is needed to prove the next proposition.

Lemma 7.1. *There exists $c > 0$ such that for every $h \in W^{1,\infty}(\mathbb{R}^2)$,*

$$(7.1) \quad (G[h]f^b, f^b) \geq c(1 + \|h\|_{W^{1,\infty}(\mathbb{R}^2)})^{-2} \left\| \frac{|\nabla|}{(1 + |\nabla|)^{1/2}} f^b \right\|_{L^2(\mathbb{R}^2)}^2, \quad \forall f^b \in H^{\frac{1}{2}}(\mathbb{R}^2),$$

where $G[h]f^b$ means Dirichlet-Neumann operator,

$$G[h]f^b = (\nabla f)^b \cdot \mathbf{N}, \quad \mathbf{N} = (-\nabla h, 1), \quad \Delta f = 0.$$

Proof. See proposition 3.4 of [2]. \square

We can apply above lemma for $f = p^S$, since $\Delta p^S = \Delta^\varphi q^S = 0$.

Proposition 7.2. *Assume that $\Lambda_{m,\infty}(h, v)$ and $|h|_{1,\infty}$ is bounded. Then h enjoys the following estimate for sufficiently small time t .*

$$(7.2) \quad \int_0^t |Z^{m-1} \nabla h|_{\frac{3}{2}}^2 \leq \Lambda\left(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\right) + \bar{\theta} \|Z^{m-1} h_t(t)\|_{L^2}^2 + 2\sqrt{t} \|Z^{m-1} \nabla h_t(t)\|_{L^2}^2 \\ + \Lambda_{m,\infty}(h, v) \int_0^t \|Z^{m-1} \nabla h_t\|^2 ds + \Lambda_m(h, v) \int_0^t \left(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-2,2}}^2 + \varepsilon \|\nabla v\|_{X^{m-1,1}}^2 + \varepsilon \|v\|_{X^{m-1,2}}^2 \right).$$

Proof. From kinematic boundary condition $h_t = v^b \cdot N$, we get $\partial_{tt} h = v_t^b \cdot N + v^b \cdot N_t$. We apply Z^{m-1} to this equation, where $\alpha_3 = 0$, because we are on the boundary.

$$(7.3) \quad \partial_{tt} (Z^{m-1} h) = \left(Z^{m-1} v_t^b \right) \cdot N + v_t^b \cdot \left(Z^{m-1} N \right) + [Z^{m-1}, v_t^b, N] + \left(Z^{m-1} v^b \right) \cdot N_t + v^b \cdot \left(Z^{m-1} N_t \right) + [Z^{m-1}, v^b, N_t] \\ = \left\{ -Z^{m-1} (v \cdot \nabla^\varphi v)^b - Z^{m-1} (\nabla^\varphi q^E + \nabla^\varphi q^{NS})^b + 2\varepsilon Z^{m-1} (\nabla^\varphi \cdot S^\varphi v)^b \right\} \cdot \mathbf{N} \\ + G[h] V^b + (I_1 + I_2 + I_3 + I_4 + I_5) \\ = G[h] V^b + (I_1 + I_2 + I_3 + I_4 + I_5) + (J_1 + J_2 + J_3),$$

where

$$(7.4) \quad V^b = \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right), \\ I_1 = v_t^b \cdot \left(Z^{m-1} N \right), \\ I_2 = [Z^{m-1}, v_t^b, N], \\ I_3 = \left(Z^{m-1} v^b \right) \cdot N_t, \\ I_4 = v^b \cdot \left(Z^{m-1} N_t \right), \\ I_5 = [Z^{m-1}, v^b, N_t], \\ J_1 = -Z^{m-1} (v \cdot \nabla^\varphi v)^b \cdot \mathbf{N}, \\ J_2 = -Z^{m-1} (\nabla^\varphi q^E + \nabla^\varphi q^{NS})^b \cdot \mathbf{N}, \\ J_3 = 2\varepsilon Z^{m-1} (\nabla^\varphi \cdot S^\varphi v)^b \cdot \mathbf{N}.$$

We take dot product with V^b and $\int_0^t \int_{\partial S}$ to the equation (7.3). For the left hand side,

$$(7.5) \quad \int_0^t \int_{\partial S} Z^{m-1} h_{tt} \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) = - \int_0^t \int_{\partial S} Z^{m-1} \nabla h_{tt} \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \\ = - \left[\int_{\partial S} Z^{m-1} \nabla h_t \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \right]_0^t + \int_0^t \int_{\partial S} Z^{m-1} \nabla h_t \cdot Z^{m-1} \partial_t \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \\ \leq \Lambda\left(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\right) + \int_0^t \Lambda_m(h, v) \|\nabla h_t\|_{X^{m-1,0}}^2 - \int_{\partial S} Z^{m-1} \nabla h_t \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \Big|_t.$$

Now we estimate other terms involving I_k and J_k .

1) Estimate of $\int_0^t \int_{\partial S} I_1 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) d\text{Ads}$.

$$\int_0^t \int_{\partial S} v_t^b \cdot \left(Z^{m-1} N \right) \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \leq \int_0^t \Lambda_m(h, v) |h|_{X^{m-2,2}}^2.$$

2) Estimate of $\int_0^t \int_{\partial S} I_2 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) d\text{Ads}$.

$$\int_0^t \int_{\partial S} [Z^{m-1}, v_t^b, N] \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \leq \int_0^t \Lambda_m(h, v) \left(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-2,2}}^2 \right).$$

We used trace estimate Lemma 2.3 here.

3) Estimate of $\int_0^t \int_{\partial S} I_3 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$.

$$\int_0^t \int_{\partial S} (Z^{m-1} v^b) \cdot N_t \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \leq \int_0^t \Lambda_m(h, v) \left(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-2,2}}^2 \right),$$

similarly as above.

4) Estimate of $\int_0^t \int_{\partial S} I_4 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$.

$$\int_0^t \int_{\partial S} v^b \cdot (Z^{m-1} N_t) \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \leq \int_0^t \Lambda_m(h, v) |h|_{X^{m-2,2}}^2.$$

5) Estimate of $\int_0^t \int_{\partial S} I_5 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$.

$$\int_0^t \int_{\partial S} [Z^{m-1}, v_t^b, N] \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \leq \int_0^t \Lambda_m(h, v) \left(\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_{X^{m-2,2}}^2 \right).$$

6) Estimate of $\int_0^t \int_{\partial S} J_1 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$. Using Lemma 2.3 we can replace v^b into v by giving $|\cdot|_{H^{-1/2}}$. Hence,

$$\begin{aligned} \int_0^t \int_{\partial S} -Z^{m-1} (v \cdot \nabla^\varphi v)^b \cdot \mathbf{N} \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) &\leq \int_0^t \left| Z^{m-1} (v \cdot \nabla^\varphi v)^b \right|_{-\frac{1}{2}} \left| \mathbf{N} \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \right|_{\frac{1}{2}} ds \\ &\leq \int_0^t \Lambda_m(h, v) (\|v\|_{X^{m-1,1}} + \|\nabla v\|_{X^{m-1,0}}) |h|_{X^{m-1, \frac{5}{2}}} \\ &\leq \int_0^t \Lambda_m(h, v) (\|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-1,0}}^2) + \theta \int_0^t \Lambda_m(h, v) |\nabla h|_{X^{m-1, \frac{3}{2}}}^2, \end{aligned}$$

where θ is sufficiently small because we used Young's inequality.

7) Estimate of $\int_0^t \int_{\partial S} J_2 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$. We perform estimate same as J_1 case. Hence, using pressure estimate in section 4,

$$\begin{aligned} \int_0^t \int_{\partial S} -Z^{m-1} (\nabla^\varphi q^E + \nabla^\varphi q^{NS})^b \cdot \mathbf{N} \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \\ \leq \int_0^t \Lambda_m(h, v) (|h|_{X^{m-1,2}}^2 + \|v\|_{X^{m-1,1}}^2 + \varepsilon \|\nabla v\|_{X^{m-1,1}}^2 + \varepsilon \|v\|_{X^{m-1,2}}^2) + \theta \int_0^t \Lambda_m(h, v) |\nabla h|_{X^{m-1, \frac{3}{2}}}^2. \end{aligned}$$

8) Estimate of $\int_0^t \int_{\partial S} J_3 \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) dA ds$. Similarly as above estimate, we get

$$\begin{aligned} \int_0^t \int_{\partial S} 2\varepsilon Z^{m-1} (\nabla^\varphi \cdot S^\varphi v)^b \cdot \mathbf{N} \nabla \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \\ \leq \int_0^t \Lambda_m(h, v) (\varepsilon \|\nabla v\|_{X^{m-1,1}}^2 + \varepsilon \|v\|_{X^{m-1,2}}^2) + \theta \int_0^t \Lambda_m(h, v) |\nabla h|_{X^{m-1, \frac{3}{2}}}^2. \end{aligned}$$

9) Estimate of $\int_{\partial S} Z^{m-1} \nabla h_t \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \Big|_t$.

$$\begin{aligned} &\int_{\partial S} Z^{m-1} \nabla h_t \cdot Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \Big|_t \\ &= \int_{\partial S} Z^{m-1} \nabla h_t(t) \left\{ Z^{m-1} \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) (0) + \int_0^t Z^{m-1} \partial_t \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) ds \right\} \\ &= - \int_{\partial S} Z^{m-1} h_t(t) \nabla \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) (0) + \int_{\partial S} Z^{m-1} \nabla h_t(t) \int_0^t Z^{m-1} \partial_t \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) ds \\ &\leq \Lambda \left(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}} \right) + \bar{\theta} \|Z^{m-1} h_t(t)\|_{L^2}^2 + \sqrt{t} \|Z^{m-1} \nabla h_t(t)\|_{L^2} \left\{ \int_0^t \left\| Z^{m-1} \partial_t \left(\frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \right\|_{L^2}^2 ds \right\}^{1/2} \end{aligned}$$

$$\leq \Lambda\left(\frac{1}{c_0}, \|h(0)\|_{X^{m,1}}\right) + \bar{\theta}\|Z^{m-1}h_t(t)\|_{L^2}^2 + 2\sqrt{t}\|Z^{m-1}\nabla h_t(t)\|_{L^2}^2 + \sqrt{t}\Lambda_{m,\infty}(h, v) \int_0^t \|Z^{m-1}\nabla h_t(t)\|_{L^2}^2 ds,$$

where sufficiently small $\bar{\theta}$ came from Young's inequality.

10) Estimate of Dirichlet-Neumann operator term.

$$\left(1 + |h|_{W^{1,\infty}}\right)^{-2} \left| \frac{|\nabla|}{(1 + |\nabla|)^{1/2}} V^b \right|_{L^2}^2 \leq (G[h]V^b, V^b).$$

Note that, (\mathcal{F} means Fourier Transform with respect to horizontal direction.)

$$\begin{aligned} (7.6) \quad & \left| \frac{|\nabla|}{(1 + |\nabla|)^{1/2}} V^b \right|_{L^2(\partial S)} \geq \left| \frac{|\xi|^2}{(1 + |\xi|)^{1/2}} \mathcal{F}\left(Z^{m-1} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) \right|_{L^2(\partial S)} \\ & = \left| \left(\frac{1 + |\xi|^2}{(1 + |\xi|)^{1/2}} - \frac{1}{(1 + |\xi|)^{1/2}} \right) \mathcal{F}\left(Z^{m-1} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) \right|_{L^2(\partial S)} \\ & \geq \left| Z^{m-1} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{H^{\frac{3}{2}}(\partial S)} - \left| Z^{m-1} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{H^{\frac{1}{2}}(\partial S)}. \end{aligned}$$

For $\left| Z^{m-1} \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{H^{\frac{3}{2}}(\partial S)}$, we consider the highest order terms and low order terms to get,

$$(7.7) \quad \Lambda\left(\frac{1}{c_0}\right) |\nabla h|_{X^{m-1, \frac{3}{2}}}^2 \leq \left| \frac{|\nabla|}{(1 + |\nabla|)^{1/2}} V^b \right|_{L^2(\partial S)}^2 + \text{low order terms}.$$

We collect (7.5), (7.7), and estimates 1) - 10) with sufficiently small θ, t , to finish the proof. \square

8. NORMAL DERIVATIVE ESTIMATE

From above energy estimate, we should control $\|\partial_z v\|_{X^{m-1,0}}$. But it is hard to estimate $\partial_z v$ directly. Instead we estimate S_n , which is tangential part of $S^\varphi v \mathbf{n}$.

$$(8.1) \quad S_n = \Pi(S^\varphi v \mathbf{n}) \quad \text{where} \quad \Pi = I - \mathbf{n} \otimes \mathbf{n}.$$

First, we show that instead of $\partial_z v$, we are suffice to estimate S_n

Lemma 8.1. *We have the following normal part estimate of $\partial_z v$.*

$$(8.2) \quad \|\partial_z v \cdot \mathbf{n}\|_{X^{m-1,0}} \leq \Lambda_{m,\infty}(h, v) \left(|h|_{X^{m-1,1}} + \|v\|_{X^{m-1,1}} \right).$$

Proof. From divergence free condition, we have,

$$(8.3) \quad \partial_z v \cdot \mathbf{n} = \frac{1}{\sqrt{1 + |\nabla_y \varphi|^2}} \partial_z \varphi (\partial_1 v_1 + \partial_2 v_2).$$

Applying Z^{m-1} and using basic propositions, we easily get

$$(8.4) \quad \left\| Z^{m-1} (\partial_z v \cdot \mathbf{n}) \right\| \leq \Lambda_{m,\infty}(h, v) \left(|h|_{X^{m-1,1}} + \|v\|_{X^{m-1,1}} \right).$$

\square

Using this lemma, we can estimate $\partial_z v$.

Lemma 8.2. *For L^2 type norm, $\partial_z v$ can be controlled by S_n . i.e*

$$(8.5) \quad \|\partial_z v\|_{X^{m-1,0}} \leq \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}} \right).$$

Proof.

$$(8.6) \quad 2S^\varphi v \mathbf{n} = (\nabla u) \mathbf{n} + (\nabla u)^T \mathbf{n} = (\nabla u) \mathbf{n} + g^{ij} (\partial_j v \cdot \mathbf{n}) \partial_{y^i}.$$

And from divergence free condition,

$$(8.7) \quad \partial_N u = \frac{1 + |\nabla_y \varphi|^2}{\partial_z \varphi} \partial_z v - \partial_1 \varphi \partial_1 v - \partial_2 \varphi \partial_2 v$$

to obtain

$$(8.8) \quad \|\partial_z v\|_{X^{m-1,0}} \leq \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|\partial_z v \cdot \mathbf{n}\|_{X^{m-1,0}} + \|S^\varphi v \mathbf{n}\|_{X^{m,0}} \right)$$

and

$$(8.9) \quad S^\varphi v \mathbf{n} = S_n + (\mathbf{n} \otimes \mathbf{n}) (S^\varphi v \mathbf{n}).$$

Now we use previous lemma to get

$$(8.10) \quad \|\partial_z v\|_{X^{m-1,0}} \leq \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}} \right).$$

□

Now we estimate S_n . As like in [1], we take ∇^φ to the navier-stokes equation.

$$(8.11) \quad \partial_t^\varphi \nabla^\varphi v + (v \cdot \nabla^\varphi) \nabla^\varphi v + (\nabla^\varphi v)^2 + (D^\varphi)^2 q - \varepsilon \Delta^\varphi \nabla^\varphi v = 0,$$

where $(D^\varphi)^2$ is Hessian matrix. We also take symmetric part of the equation, then using both equations,

$$(8.12) \quad \partial_t^\varphi S^\varphi v + (v \cdot \nabla^\varphi) S^\varphi v + \frac{1}{2} \left((\nabla v)^2 + ((\nabla v)^T)^2 \right) + (D^\varphi)^2 q - \varepsilon \Delta^\varphi (S^\varphi v) = 0.$$

By taking tangential operator, Π ,

$$(8.13) \quad \partial_t^\varphi S_n + (v \cdot \nabla^\varphi) S_n - \varepsilon \Delta^\varphi (S_n) = F_S,$$

where F_S is commutator,

$$(8.14) \quad F_S = F_S^1 + F_S^2 + F_S^3,$$

$$(8.15) \quad \begin{aligned} F_S^1 &= -\frac{1}{2} \Pi \left((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^T)^2 \right) \mathbf{n} + (\partial_t \Pi + v \cdot \nabla^\varphi \Pi) S^\varphi v \mathbf{n} + \Pi S^\varphi v (\partial_t \mathbf{n} + v \cdot \nabla^\varphi \mathbf{n}), \\ F_S^2 &= -2\varepsilon \partial_i^\varphi \Pi \partial_i^\varphi (S^\varphi v \mathbf{n}) - 2\varepsilon \Pi (\partial_i^\varphi (S^\varphi v) \partial_i^\varphi \mathbf{n}) - \varepsilon (\Delta^\varphi \Pi) S^\varphi v \mathbf{n} - \varepsilon \Pi S^\varphi v \Delta^\varphi \mathbf{n}, \\ F_S^3 &= -\Pi \left((D^\varphi)^2 q \right) \mathbf{n}. \end{aligned}$$

We will apply Z^{m-1} to the equation, so we need to estimate $\|F_S^1\|_{X^{m-1,0}}, \|F_S^2\|_{X^{m-1,0}}, \|F_S^3\|_{X^{m-1,0}}$. In [1], optimal estimate order was $m-2$. This is because,

$$\|F_S^3\|_{m-2} \sim \|D^2 q\|_{m-2} \sim |h|_m,$$

which is optimal for regularity of h . In our case h has better regularity but q is worse. Roughly,

$$\|F_S^3\|_{m-2} \sim \|D^2 q\|_{m-2} \sim |h|_{m+\frac{3}{2}}.$$

This means for surface tension case, optimal regularity for S_n is $m-3$, which is even worse than non-surface tension case. For $Z^{m-3} F_S^1$, using propositions and lemmas in section 2,

$$(8.16) \quad \begin{aligned} \|Z^{m-3} F_S^1\|_{L^2(S)} &\leq \Lambda_{m,\infty}(h, v) \left(\|\nabla v\|_{X^{m-3,0}} + |h|_{X^{m-3,1}} + \|v\|_{X^{m-3,0}} \right) \\ &\leq \Lambda_{m,\infty}(h, v) \left(\|S_n\|_{X^{m-3,0}} + |h|_{X^{m-1,1}} + \|v\|_{X^{m,0}} \right). \end{aligned}$$

Similary, for $Z^{m-3} F_S^2$,

$$(8.17) \quad \|Z^{m-3} F_S^2\|_{L^2(S)} \leq \varepsilon \Lambda_{m,\infty}(h, v) \left(\|\partial_{zz} v\|_{X^{m-3,0}} + \|\partial_z v\|_{X^{m-2,0}} + |h|_{X^{m-3,\frac{5}{2}}} \right).$$

By using Young's inequality,

$$(8.18) \quad ab \leq \frac{a^2}{2\delta} + \frac{\delta b^2}{2}, \quad \forall \delta > 0.$$

$\varepsilon \int_S \|\nabla S_n\|_{X^{m-3,0}}$ can be absorbed in energy. For $|h|_{X^{m-3,\frac{1}{2}}}$, it can be controlled by $|h|_{X^{m-2,1}}$, by Dirichlet-Neumann operator estimate.

For $Z^{m-3}F_S^3$,

$$(8.19) \quad \|Z^{m-3}F_S^3\|_{L^2(S)} \leq \Lambda_{m,\infty}(h, v) \left(\|\nabla v\|_{X^{m-3,0}} + |h|_{X^{m,\frac{1}{2}}} + \|v\|_{X^{m-3,0}} \right).$$

Now we apply Z^α , with $|\alpha| = m-3$ to get,

$$(8.20) \quad \partial_t^\varphi Z^\alpha S_n + (v \cdot \nabla^\varphi) Z^\alpha S_n - \varepsilon \Delta^\varphi Z^\alpha S_n = Z^\alpha (F_S) + C_S,$$

where C_S is commutator. As like in [1], we divide C_S into,

$$(8.21) \quad C_S^1 = [Z^\alpha v_y] \cdot \nabla_y S_n + [Z^\alpha, V_z] \partial_z S_n \doteq C_{S_y} + C_{S_z}, \quad C_S^2 = -\varepsilon [Z^\alpha, \Delta^\varphi] S_n.$$

Since $(Z^\alpha S_n)_{z=0} = 0$, we get the following.

$$(8.22) \quad \frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n|^2 dV_t + \varepsilon \int_S |\nabla^\varphi Z^\alpha S_n|^2 dV_t = \int_S (Z^\alpha F_S + C_S) \cdot Z^\alpha S_n dV_t.$$

Estimate of C_{S_y} is easy. we get,

$$(8.23) \quad \|C_{S_y}\| \leq \Lambda_{m,\infty}(h, v) \left(\|S_n\|_{X^{m-3,0}} + \|v\|_{X^{m-3,0}} + \|\partial_z v\|_{X^{m-4,0}} \right).$$

To estimate C_{S_z} is not easy, because it contains C_{S_z} , which is not controlled yet. We give ∂_z to V_z by integration by part. From the commutator, we have to control the terms like,

$$(8.24) \quad \|Z^\beta V_z \partial_z Z^\gamma S_n\|,$$

where $|\beta| + |\gamma| \leq m-3$, $|\gamma| \leq m-4$ or equivalently $|\beta| \neq 0$. We interchange ∂_z and Z_3 by

$$(8.25) \quad Z^\beta V_z \partial_z Z^\gamma S_n = \frac{1-z}{z} Z^\beta V_z Z_3 Z^\gamma S_n,$$

then by commutation between $\frac{1-z}{z}$ and Z^β , we encounter the terms like this, where $c_{\tilde{\beta}}$ is some nice, bounded function and $|\tilde{\beta}| \leq |\beta|$.

$$(8.26) \quad c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1-z}{z} V_z \right) Z_3 Z^\gamma S_n.$$

If $\tilde{\beta} = 0$,

$$(8.27) \quad \left\| c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1-z}{z} V_z \right) Z_3 Z^\gamma S_n \right\| \lesssim \|S_n\|_{X^{m-3,0}}.$$

If $\tilde{\beta} \neq 0$,

$$(8.28) \quad \left\| c_{\tilde{\beta}} Z^{\tilde{\beta}} \left(\frac{1-z}{z} V_z \right) Z_3 Z^\gamma S_n \right\| \lesssim \left\| Z \left(\frac{1-z}{z} V_z \right) \right\|_{Y^{\frac{m}{2},0}} \|S_n\|_{X^{m-3,0}} + \|S_n\|_{Y^{\frac{m}{2},0}} \left\| Z \left(\frac{1-z}{z} V_z \right) \right\|_{X^{m-4,0}}.$$

First, we see that,

$$(8.29) \quad \left\| Z \left(\frac{1-z}{z} V_z \right) \right\|_{Y^{\frac{m}{2},0}} \lesssim \|V_z\|_{Y^{\frac{m}{2}+1,0}} + \|\partial_z V_z\|_{Y^{\frac{m}{2}+1,0}}$$

and,

$$(8.30) \quad \left\| Z \left(\frac{1-z}{z} V_z \right) \right\|_{X^{m-4,0}} \lesssim \|\nabla V_z\|_{X^{m-4,0}} + \left\| \frac{1}{z(1-z)} V_z \right\|_{X^{m-4,0}} = \left\| \frac{1-z}{z} Z V_z \right\|_{X^{m-4,0}} + \left\| \frac{1}{z(1-z)} V_z \right\|_{X^{m-4,0}}.$$

So we should estimate the terms that look like,

$$(8.31) \quad \left\| \frac{1-z}{z} Z^\xi Z V_z \right\|, \quad \left\| \frac{1}{z(1-z)} Z^\xi V_z \right\|,$$

where $|\xi| \leq m-4$. To estimate these two types of terms, we use the following lemma.

Lemma 8.3. *If $f(0) = 0$, we have the following inequalities,*

$$(8.32) \quad \int_{-\infty}^0 \frac{1}{z^2(1-z)^2} |f(z)|^2 dz \lesssim \int_{-\infty}^0 |\partial_z f(z)|^2 dz,$$

$$(8.33) \quad \int_{-\infty}^0 \left(\frac{1-z}{z} \right)^2 |f(z)|^2 dz \lesssim \int_{-\infty}^0 \left(|f(z)|^2 + |\partial_z f(z)|^2 \right) dz.$$

Proof. See [1], Lemma 8.4 □

Using above lemma, we have

$$(8.34) \quad \left\| \frac{1-z}{z} Z^\xi Z V_z \right\|^2 \lesssim \|Z^\xi Z V_z\|^2 + \left\| \partial_z \left(Z^\xi Z V_z \right) \right\|^2,$$

$$(8.35) \quad \left\| \frac{1}{z(1-z)} Z^\xi V_z \right\| \lesssim \|\partial_z Z^\xi V_z\|.$$

So,

$$(8.36) \quad \|C_{S_z}\| \lesssim \|Z V_z\|_{X^{m-4,0}} + \|\partial_z Z V_z\|_{X^{m-4,0}} + \|\partial_z V_z\|_{X^{m-4,0}}$$

Combining with C_{S_y} , we have

$$(8.37) \quad \begin{aligned} \|C_S^1\| &\leq \Lambda_{m,\infty}(h, v) \left(\|S_n\|_{X^{m-3,0}} + \|v\|_{X^{m-3,0}} + \|\partial_z V_z\|_{X^{m-3,0}} \right) \\ &\leq \Lambda_{m,\infty}(h, v) \left(\|S_n\|_{X^{m-3,0}} + \|v\|_{X^{m-2,0}} + |h|_{X^{m-3,1}} \right). \end{aligned}$$

For C_S^2 , we have

$$(8.38) \quad \begin{aligned} \varepsilon Z^\alpha (\Delta^\varphi S_n) &= \varepsilon Z^\alpha \left(\frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla S_n) \right) = \varepsilon \frac{1}{\partial_z \varphi} Z^\alpha \left(\nabla \cdot (E \nabla S_n) \right) + C_{S_1}^2 \\ &= \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot Z^\alpha (E \nabla S_n) + C_{S_2}^2 + C_{S_1}^2 \\ &= \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot (E \nabla Z^\alpha S_n) + C_{S_3}^2 + C_{S_2}^2 + C_{S_1}^2 \\ &= \varepsilon \Delta^\varphi (Z^\alpha S_n) + C_S^2 \end{aligned}$$

So, we define

$$(8.39) \quad C_S^2 = C_{S_1}^2 + C_{S_2}^2 + C_{S_3}^2,$$

where

$$(8.40) \quad C_{S_1}^2 \doteq \varepsilon \left[Z^\beta, \frac{1}{\partial_z \varphi} \right] \nabla \cdot (E \nabla S_n), \quad C_{S_2}^2 \doteq \varepsilon \frac{1}{\partial_z \varphi} [Z^\alpha, \nabla] \cdot (E \nabla S_n), \quad C_{S_3}^2 \doteq \varepsilon \frac{1}{\partial_z \varphi} \nabla \cdot ([Z^\alpha, E \nabla] S_n).$$

1) $C_{S_1}^2$

We need to estimate like,

$$(8.41) \quad \varepsilon \int_S Z^\beta \left(\frac{1}{\partial_z \varphi} \right) Z^{\tilde{\gamma}} \left(\nabla \cdot (E \nabla S_n) \right) \cdot Z^\alpha S_n dV_t,$$

where $|\beta| + |\tilde{\gamma}| = \alpha$, $\beta \neq 0$. Then again by commutator between $Z^{\tilde{\gamma}}$ and ∇ , the forms becomes like the following forms.

$$(8.42) \quad \varepsilon \int_S Z^\beta \left(\frac{1}{\partial_z \varphi} \right) \partial_i Z^\gamma \left((E \nabla S_n)_j \right) \cdot Z^\alpha S_n dV_t,$$

where $|\tilde{\gamma}| \leq |\gamma|$. Now we preform integrate by part, to get

$$(8.43) \quad \begin{aligned} &\left| \varepsilon \int_S Z^\beta \left(\frac{1}{\partial_z \varphi} \right) \partial_i Z^\gamma \left((E \nabla S_n)_j \right) \cdot Z^\alpha S_n dV_t \right| \\ &\leq \left| \varepsilon \int_S \partial_i Z^\beta \left(\frac{1}{\partial_z \varphi} \right) Z^\gamma \left((E \nabla S_n)_j \right) \cdot Z^\alpha S_n dV_t \right| + \left| \varepsilon \int_S Z^\beta \left(\frac{1}{\partial_z \varphi} \right) Z^\gamma \left((E \nabla S_n)_j \right) \cdot \partial_i Z^\alpha S_n dV_t \right|. \end{aligned}$$

Using basic propositions and dividing each terms into L^∞ , L^2 , L^2 , then we get,

$$(8.44) \quad \left| \int_S C_{S_1}^2 \cdot Z^\alpha S_n dV_t \right| \lesssim \varepsilon \Lambda_{m,\infty}(h, v) \left(\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-3,0}} + |h|_{X^{m-2,1}} \right).$$

2) $C_{S_2}^2$

We need to estimate terms such as,

$$(8.45) \quad \varepsilon \int_S \partial_z Z^\beta (E \nabla S_n) \cdot Z^\alpha S_n dV_t,$$

where $\beta \leq m-4$. Then again by integration by parts, we can get the same estimate as like $C_{S_1}^2$

$$(8.46) \quad \left| \int_S C_{S_2}^2 \cdot Z^\alpha S_n dV_t \right| \lesssim \varepsilon \Lambda_{m,\infty}(h, v) \left(\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-3,0}} + |h|_{X^{m-2,1}} \right).$$

3) $C_{S_3}^2$

We give $\nabla \cdot$ to $Z^\alpha S_n$ by integration by parts, then easily,

$$(8.47) \quad \left| \int_S C_{S_3}^2 \cdot Z^\alpha S_n dV_t \right| \lesssim \varepsilon \|[Z^\alpha, E \nabla] S_n\| \|\nabla Z^\alpha S_n\| \lesssim \varepsilon \Lambda_{m,\infty}(h, v) \left(\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-3,0}} + |h|_{X^{m-2,1}} \right).$$

Combining above three estimates, we have the following.

$$(8.48) \quad \left| \int_S C_S^2 \cdot Z^\alpha S_n dV_t \right| \leq \varepsilon \Lambda_{m,\infty}(h, v) \left(\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-3,0}} + |h|_{X^{m-2,1}} \right).$$

Combining this and estimate for F_S ,

$$(8.49) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n|^2 dV_t + \frac{\varepsilon}{2} \int_S |\nabla^\varphi Z^\alpha S_n|^2 dV_t \\ & \leq \Lambda_{m,\infty}(h, v) \left\{ \|S_n\|_{X^{m-3,0}} + |h|_{X^{m-3,1}} + \|v\|_{X^{m-2,0}} + \varepsilon \left(\|\nabla S_n\|_{X^{m-3,0}} + |h|_{X^{m-3,5/2}} + |h|_{X^{m-2,1}} \right) \right\}. \end{aligned}$$

Now we integrate for time and sum for all indices of α , and then use Young's inequality (take δ sufficiently small if needed.) to make dissipation on right hand side is absorbed by left hand side. Then we can get the following result.

Proposition 8.4. *From above argument, we have the following L^2 type estimate for S_n .*

$$(8.50) \quad \begin{aligned} & \|S_n(t)\|_{X^{m-3,0}}^2 + \varepsilon \int_0^t \|\nabla S_n(s)\|_{X^{m-3,0}}^2 ds \\ & \leq \Lambda_0 \|S_n(0)\|_{X^{m-3,0}}^2 + \int_0^t \Lambda_{m,\infty}(h, v) \left(\|S_n\|_{X^{m-3,0}} + \|v\|_{X^{m-2,0}} + |h|_{X^{m-2,1}} + |h|_{X^{m-3,5/2}} \right) ds. \end{aligned}$$

9. L^∞ TYPE ESTIMATE

In the previous section, we estimated L^2 -type norm of $\partial_z v$. We also should estimate L^∞ -type norms, like $\partial_z v$, $\|\nabla v\|_{Y^{\frac{m}{2},0}}$ to control $\Lambda_{m,\infty}$. Again, instead of $\partial_z v$, we estimate S_n . Note that L^∞ estimate for h is trivial by standard sobolev embedding.

Lemma 9.1. *We have the following estimate for normal part of $\partial_z v$.*

$$(9.1) \quad \|\partial_z v \cdot n\|_{Y^{k,0}} \lesssim \Lambda \left(\frac{1}{c_0}, |h|_{Y^{k,0}} \right) \|v\|_{X^{k+1,0}}.$$

Proof. From divergence free condition,

$$(9.2) \quad \partial_z v \cdot N = \partial_z \varphi (\partial_1 v_1 + \partial_2 v_2).$$

We take $\|\cdot\|_{Y^{k,0}}$ so get the result. □

Similar to the previous section, we use

$$2S^\varphi v \mathbf{n} = (\nabla u) \mathbf{n} + (\nabla u)^T \mathbf{n} = (\nabla u) \mathbf{n} + g^{ij} (\partial_j v \cdot \mathbf{n}) \partial_{y^i}$$

and divergence free condition,

$$\partial_N u = \frac{1 + |\nabla_y \varphi|^2}{\partial_z \varphi} \partial_z v - \partial_1 \varphi \partial_1 v - \partial_2 \varphi \partial_2 v..$$

So we obtain,

$$(9.3) \quad \|\partial_z v\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{k,0}}\right) \left(\|v\|_{Y^{k+1,0}} + \|S^\varphi v \mathbf{n}\|_{Y^{k,0}} \right)$$

and since

$$S^\varphi v \mathbf{n} = S_n + (\mathbf{n} \otimes \mathbf{n}) (S^\varphi v \mathbf{n}),$$

$$(9.4) \quad \|S^\varphi v \mathbf{n}\| \lesssim \|S_n\| + \Lambda\left(\frac{1}{c_0}, |h|_{Y^{k,0}}\right) \|\partial_z v \cdot \mathbf{n}\|_{Y^{k,0}} + \|v\|_{Y^{k,0}},$$

by using above lemma for normal part of $\partial_z v$,

$$\lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{k,0}}\right) \left(\|v\|_{Y^{k+1,0}} + \|S_n\|_{Y^{k,0}} \right).$$

We use anisotropic embedding to $\|v\|_{Y^{k+1,0}}$.

$$\|v\|_{Y^{k+1,0}}^2 \lesssim \|\partial_z v\|_{X^{k+2,0}} \|v\|_{X^{k+3,0}}.$$

Hence, we get the following proposition.

Proposition 9.2. *We have the following.*

$$(9.5) \quad \|\partial_z v\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{k,0}}\right) \left(\|S_n\|_{Y^{k,0}} + \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} \right).$$

Note that for sufficiently small k , (than m), then $\partial_z v$ and S_n are equivalent in L^∞ -type norm.

Above proposition implies we are suffice to estimate $\|S_n\|_{Y^{\frac{m}{2},0}}$, instead of $\|\partial_z v\|_{Y^{\frac{m}{2},0}}$. So as we did in previous section, we use equation for S_n with Dirichlet boundary condition. Main difficulty in this section is commutator between $Z_3 = \frac{z}{1-z} \partial_z$ and Δ^φ . This commutator was not a problem in basic L^2 -type energy estimate of v and S_n , because the highest order commutator, which looks like $\sim \varepsilon Z^\alpha \partial_z S_n$ can be absorbed into dissipation term in the energy. But, in L^∞ estimate, we use the following maximal principle for convection-diffusion equation. That is for equation for S_n ,

$$(9.6) \quad \partial_t^\varphi S_n + (v \cdot \nabla^\varphi) S_n - \varepsilon \Delta^\varphi S_n = F_S.$$

We have the following L^∞ -type estimate which does not have dissipation in energy term.

$$(9.7) \quad \|S_n(t)\|_{L^\infty} \leq \|S_n(0)\|_{L^\infty} + \int_0^t \|F_S(s)\|_{L^\infty} ds.$$

So, if we have the commutator which have 1-more derivative than $Z^\alpha S_n$, we cannot control them with energy, although it has ε as its coefficient. Note that, L^∞ terms cannot be controlled by Sobolev co-normal space. That means, standard Sobolev embedding does not hold for co-normal space in general. This is because of behavior of near the boundary. But, away from the boundary $\frac{z}{1-z}$ is not zero, and its all order derivative for z is always uniformly bounded. Now, we divided co-normal function into two parts, one is supported near the boundary and another is supported away from boundary. Then 2nd stuff are easy to be controlled by Sobolev embedding. For the first stuff, we deform the coordinate so that locally ∂_{zz}^φ look like ∂_{zz} . Then ∂_z commute with ∂_{zz} , so it does not generate any harmful (which has 1-more order than L^∞ -type energy) commutator. This clever idea is introduced in [15] (and also in [1]). We introduce this system briefly and use similar arguments to get the result. First, we start with very simple lemma, which means away from the boundary Sobolev co-normal is just like standard Sobolev.

Lemma 9.3. *For any smooth cut-off function $\bar{\chi}$ such that $\bar{\chi} = 0$ in a vicinity of $z = 0$, we have for $m > k + 3/2$:*

$$(9.8) \quad \|\bar{\chi} f\|_{W^{k,\infty}} \lesssim \|f\|_{H_{co}^m}.$$

Now we decompose $Z^k S_n$ as

$$\|Z^k S_n\|_{L^\infty} \leq \|\chi Z^k S_n\|_{L^\infty} + \|v\|_{Y^{k+1,0}}.$$

To estimate $\|v\|_{Y^{k+1,0}}$, we use the following proposition.

Proposition 9.4. *We have the following estimate.*

$$(9.9) \quad \|v\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|\nabla v\|_{Y^{1,0}} + \|v\|_{X^{k+2,0}} + |h|_{X^{k+1,1}} + \|S_n\|_{X^{k+1,0}}\right).$$

Proof. Using anisotropic Sobolev embedding,

$$\|Z^k v\|_{L^\infty}^2 \lesssim \|\partial_z v\|_{X^{k+1,0}} \|v\|_{X^{k+2,0}}$$

and using lemma of previous section,

$$\|\partial_z v\|_{X^{k+1,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{k+2}{2},0}}\right) \left(\|v\|_{X^{k+2,0}} + |h|_{X^{k+1,1}} + \|S_n\|_{X^{k+1,0}}\right).$$

Then using induction for $\|\nabla v\|_{Y^{\frac{k+2}{2},0}}$, until it become 1. And notice that $\|v\|_{X^{k+2,0}}$ is absorbed by estimate of $\|\partial_z v\|_{X^{k+1,0}}$. \square

See that above proposition means that we are suffice to estimate $Z^k S_n$ only near the boundary, so now we introduce modified coordinate which was introduced in [15] and [1]. Let, define transformation Ψ ,

$$(9.10) \quad \begin{aligned} \Psi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) &\rightarrow \Omega_t, \\ x = (y, z) &\mapsto \begin{pmatrix} y \\ h(t, y) \end{pmatrix} + z \mathbf{n}^b(t, y), \end{aligned}$$

where \mathbf{n}^b is unit normal at the boundary, $(-\nabla h, 1)/|N|$. To show that this is diffeomorphism near the boundary, we check

$$D\Psi(t, \cdot) = \begin{pmatrix} 1 & 0 & -\partial_1 h \\ 0 & 1 & -\partial_2 h \\ \partial_1 h & \partial_2 h & 1 \end{pmatrix} + \begin{pmatrix} -z\partial_{11}h & -z\partial_{12}h & 0 \\ -z\partial_{21}h & -z\partial_{22}h & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is diffeomorphism near the boundary since norm of second matrix is controlled by $|h|_{2,\infty}$. So, we restrict $\Psi(t, \cdot)$ on $\mathbb{R}^2 \times (-\delta, 0)$ so that it is diffeomorphism. (δ is depend on c_0). Of course, think that above support separation was done by $\chi(z) = \kappa(\frac{z}{\delta(c_0)})$. Now we write laplacian Δ^φ with respect to Riemannian metric of above parametrization. Riemannian metric becomes,

$$(9.11) \quad g(y, z) = \begin{pmatrix} \tilde{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix},$$

where \tilde{g} is 2×2 block matrix. And with this metric, laplacian becomes,

$$(9.12) \quad \Delta_g f = \partial_{zz} f + \frac{1}{2} \partial_z (\ln |g|) \partial_z f + \Delta_{\tilde{g}} f,$$

where

$$(9.13) \quad \Delta_{\tilde{g}} f = \frac{1}{|\tilde{g}|^{\frac{1}{2}}} \sum_{1 \leq i, j \leq 2} \partial_{y^i} (\tilde{g}^{ij} |\tilde{g}|^{\frac{1}{2}} \partial_{y^j} f),$$

where \tilde{g}^{ij} is inverse matrix element of \tilde{g} . We now solve problem in domain of Ψ . We restrict $S^\varphi v$ near the boundary and parametrize them via Ψ . Let

$$(9.14) \quad S^\chi = \chi(z) S^\varphi v,$$

where $\chi(z) = \kappa(\frac{z}{\delta(c_0)}) \in [0, 1]$ where κ is smooth and compactly supported near the boundary, taking value 1 there. Equation for S^χ is

$$(9.15) \quad \partial_t^\varphi S^\chi + (v \cdot \nabla^\varphi) S^\chi - \varepsilon \Delta^\varphi S^\chi = F_{S^\chi},$$

where

$$\begin{aligned} F_{S^\chi} &= F^\chi + F_v, \\ F^\chi &= (V_z \partial_z \chi) S^\varphi v - \varepsilon \nabla^\varphi \chi \cdot \nabla^\varphi S^\varphi v - \varepsilon \Delta^\varphi \chi S^\varphi v, \end{aligned}$$

$$F_v = -\chi(D^\varphi)^2 q - \frac{\chi}{2}((\nabla^\varphi v)^2 + ((\nabla^\varphi v)^t)^2).$$

Note that F^χ is supported away from the boundary. We rewrite this function on our new frame by taking $\Phi^{-1} \circ \Psi$. We define

$$(9.16) \quad S^\Psi(t, y, z) = S^\chi(t, \Phi^{-1} \circ \Psi)$$

and S^Ψ solves

$$(9.17) \quad \partial_t S^\Psi + w \cdot \nabla S^\Psi - \varepsilon(\partial_{zz} S^\Psi + \frac{1}{2} \partial_z(\ln|g|) \partial_z S^\Psi + \Delta_{\tilde{g}} S^\Psi) = F_{S^\chi}(t, \Phi^{-1} \circ \Psi),$$

where

$$w = \bar{\chi}(D\Psi)^{-1}(v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi),$$

where $\bar{\chi}$ is slightly larger support so that $\bar{\chi} S^\Psi = S^\Psi$ and as like S^χ , S^Ψ is also only supported near the boundary. In this frame S_n correspond to S_n^Ψ , which is defined as following.

$$(9.18) \quad S_n^\Psi(t, y, z) = \Pi^b(t, y) S^\Psi \mathbf{n}^b(t, y) = \Pi^b(t, y) S^\chi(t, \Phi^{-1} \circ \Psi) \mathbf{n}^b(t, y),$$

where $\Pi^b = I - \mathbf{n}^b \otimes \mathbf{n}^b$ (tangential operator at the boundary, so they are independent to z .) Then equation for S_n^Ψ becomes,

$$(9.19) \quad \partial_t S_n^\Psi + w \cdot \nabla S_n^\Psi - \varepsilon(\partial_{zz} + \frac{1}{2} \partial_z(\ln|g|) \partial_z) S_n^\Psi = F_n^\Psi,$$

where

$$F_n^\Psi = \Pi^b F_{S^\chi} \mathbf{n}^b + F_n^{\Psi,1} + F_n^{\Psi,2},$$

where

$$\begin{aligned} F_n^{\Psi,1} &= ((\partial_t + w_y \cdot \nabla_y) \Pi^b) S^\Psi \mathbf{n}^b + \Pi^b S^\Psi (\partial_t + w_y \cdot \nabla_y) \mathbf{n}^b, \\ F_n^{\Psi,2} &= -\varepsilon \Pi^b (\Delta_{\tilde{g}} S^\Psi) \mathbf{n}^b, \end{aligned}$$

with zero-boundary condition at $z = 0$. Note that $S_n = S_n^\Psi$ on the boundary. We will estimate S_n^Ψ instead of S_n . to validate this, we should show that equivalence of these two terms. Firstly, by definition of S_n^Ψ ,

$$(9.20) \quad \|S_n^\Psi\|_{Y^{k,0}} \leq \Lambda(|h|_{Y^{k+1,0}}) \|\Pi^b S^\varphi v \mathbf{n}^b\|_{Y^{k,0}}$$

and since $|\Pi - \Pi^b| + |\mathbf{n} - \mathbf{n}^b| = O(z)$ near the boundary $z = 0$,

$$(9.21) \quad \|S_n^\Psi\|_{Y^{k,0}} \leq \Lambda\left(\frac{1}{c_0}, \|S_n\|_{Y^{k,0}} + \|v\|_{Y^{k+1,0}}\right).$$

Now, we apply anisotropic Sobolev embedding to the last term,

$$\|S_n^\Psi\|_{Y^{k,0}} \leq \Lambda\left(\frac{1}{c_0}, \|S_n\|_{Y^{k,0}} + \|\partial_z v\|_{X^{k+2,0}} + \|v\|_{X^{k+3,0}}\right).$$

For $\|\partial_z v\|_{X^{k+2,0}}$, we use Lemma 8.2 inductively, (to reduce the order of $\|\nabla v\|_{Y^{\frac{k}{2},0}}$, to get

$$(9.22) \quad \|S_n^\Psi\|_{Y^{k,0}} \leq \Lambda\left(\frac{1}{c_0}, \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} + |h|_{X^{k+2,0}} + \|S_n\|_{Y^{k,0}}\right).$$

Since we choose sufficiently smaller k than m , this estimate is okay. For opposite direction, we can do similarly to get

$$(9.23) \quad \|S_n\|_{Y^{k,0}} \leq \Lambda\left(\frac{1}{c_0}, \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} + |h|_{X^{k+2,0}} + \|S_n^\Psi\|_{Y^{k,0}}\right).$$

So, we finish equivalence argument.

Now we should apply Z^k to the system (9.17). As in [1], applying tangential derivative (Z_1, Z_2) is not that harmful, but commutator between Z_3 and Laplacian is still a problem. Critical observation in [1] is the following Lemma. (Lemma 9.6 in [1]).

Lemma 9.5. (Lemma 9.6 in [1]) Consider ρ a smooth solution of

$$(9.24) \quad \partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}, \quad z < 0, \quad \rho(t, y, 0) = 0, \quad \rho(t = 0) = \rho_0$$

for some smooth vector field w such that w_3 vanishes on the boundary. Assume that ρ and \mathcal{H} are compactly supported in z . Then we have the estimate:

$$\|Z_i \rho(t)\|_\infty \lesssim \|Z_i \rho_0\|_\infty + \|\rho_0\|_\infty + \int_0^t \left((\|w\|_{E^{2,\infty}} + \|\partial_{zz} w_3\|_{L^\infty}) (\|\rho\|_{1,\infty} + \|\rho\|_4) + \|\mathcal{H}\|_{1,\infty} \right), \quad i = 1, 2, 3.$$

We should generalize this to high order, since we need k -order L^∞ -type estimate. Let's first introduce rewriting of system (9.17), to circumvent difficulty. We set

$$(9.25) \quad \rho(t, y, z) = |g|^{\frac{1}{4}} S_n^\Psi.$$

Then ρ solves,

$$(9.26) \quad \partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = |g|^{\frac{1}{4}} (F_n^\Psi + F_g) \doteq \mathcal{H},$$

where

$$F_g = \frac{\rho}{|g|^{\frac{1}{2}}} \left(\partial_t + w \cdot \nabla - \varepsilon \partial_{zz} \right) |g|^{\frac{1}{4}},$$

which shows that $\varepsilon \partial_z \ln |g| \partial_z$ is removed. And trivially, $Z_3 S_n^\Psi$ and ρ are equivalent, i.e

$$(9.27) \quad \|\rho\|_{Y^{k,0}} \leq \Lambda(|h|_{Y^{k+1,0}}) \|S_n^\Psi\|_{Y^{k,0}}, \quad \|S_n^\Psi\|_{Y^{k,0}} \leq \Lambda(|h|_{Y^{k+1,0}}) \|\rho\|_{Y^{k,0}}.$$

Hence, instead of S_n^Ψ , we estimate ρ . Also note that equation of ρ is applicable above lemma. Now we extend above lemma to high order.

Lemma 9.6. (High order version) Consider ρ a smooth solution of

$$(9.28) \quad \partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}, \quad z < 0, \quad \rho(t, y, 0) = 0, \quad \rho(t = 0) = \rho_0$$

for some smooth vector field w such that w_3 vanishes on the boundary. Assume that ρ and \mathcal{H} are compactly supported in z . Then for any integer k , we have the following estimate:

$$\|Z^k \rho\|_{L^\infty} \lesssim \|\rho_0\|_{Y^{k,0}} + \int_0^t \left((\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}}) \|\rho\|_{X^{k+3,0}} + \|\mathcal{H}\|_{Y^{k,0}} \right) d\tau.$$

Proof. Applying co-normal derivatives to equation generate bad commutator which come from between Z_3 and Laplacian. So, we rewrite equation as

$$(9.29) \quad \partial_t \rho + z \partial_z w_3(t, y, 0) \partial_z \rho + w_y(t, y, 0) \cdot \nabla_y \rho - \varepsilon \partial_{zz} \rho = \mathcal{H} - \mathcal{R} \doteq G,$$

where ($w_3 = 0$ on the boundary)

$$\mathcal{R} = (w_y(t, y, z) - w_y(t, y, 0)) \cdot \nabla_y \rho + (w_3(t, y, z) - z \partial_z w_3(t, y, 0)) \partial_z \rho.$$

We use evolution operator $S(t, \tau)$ for homogeneous solution of above system. Let,

$$(9.30) \quad \rho(t, y, z) = S(t, \tau) f_0(y, z), \quad f(\tau, y, z) = f_0(y, z) \text{ (initial condition)}$$

solves

$$(9.31) \quad \partial_t \rho + z \partial_z w_3(t, y, 0) \partial_z \rho + w_y(t, y, 0) \cdot \nabla_y \rho - \varepsilon \partial_{zz} \rho = 0, \quad z > 0, \quad t > \tau, \quad \rho(t, y, 0) = 0.$$

For the full non-homogeneous system, by Duhamel's formula,

$$(9.32) \quad \rho(t) = S(t, 0) \rho_0 + \int_0^t S(t, \tau) G(\tau) d\tau.$$

Now, suppose that ρ is compactly supported in z , (near the boundary) and $z < 0$, then

$$\|Z_3^k \rho\|_{L^\infty} \lesssim \sum_{i,j=1}^k |z^j \partial_z^i \rho|_{L^\infty}.$$

Since z is near boundary, we don't have to consider relatively small index j , which means

$$(9.33) \quad \|Z_3^k \rho\|_{L^\infty} \lesssim \sum_{j=1}^k \sum_{i=1}^j |z^j \partial_z^i \rho|_{L^\infty}.$$

To estimate each terms on right hand side, we should control each $|z^j \partial_z^i \rho|_{L^\infty}$.

Lemma 9.7. *For evolution operator S as above, we have following estimate.*

$$(9.34) \quad |z^j \partial_z^i S(t, \tau) \rho_0|_{L^\infty} \lesssim \|\rho_0\|_{L^\infty} + \sum_{i_1+i_2=i} \|z^{j-i_1} \partial_z^{i_2} \rho_0\|_{L^\infty}.$$

Proof. Basically we follow the method of Lemma 9.6 in [1]. Let $\rho(t, y, z) = S(t, \tau) \rho_0(y, z)$ solves homogeneous system of (9.31). We extend this variables to whole space by

$$(9.35) \quad \tilde{\rho}(t, y, z) = \rho(t, y, z), \quad z > 0, \quad \tilde{\rho}(t, y, z) = -\rho(t, y, -z), \quad z < 0.$$

So that $\tilde{\rho}$ solves

$$(9.36) \quad \partial_t \tilde{\rho} + z \partial_z w_3(t, y, 0) \partial_z \tilde{\rho} + w_y(t, y, 0) \cdot \nabla_y \tilde{\rho} - \varepsilon \partial_{zz} \tilde{\rho} = 0, \quad z \in \mathbb{R},$$

with initial condition $\tilde{\rho}(\tau, y, z) = \tilde{\rho}_0(y, z)$.

By introducing \mathcal{E} , which solves,

$$\partial_t \mathcal{E} = w_y(t, \mathcal{E}, 0), \quad \mathcal{E}(\tau, \tau, y) = y$$

and define

$$g(t, y, z) = \rho(t, \mathcal{E}(t, y, z), z).$$

Then g solves,

$$(9.37) \quad \partial_t g + z \gamma(t, y) \partial_z g - \varepsilon \partial_{zz} g = 0, \quad z \in \mathbb{R}, \quad g(\tau, y, z) = \tilde{\rho}_0(y, z),$$

where

$$\gamma(t, y) = \partial_z w_3(t, \mathcal{E}(t, y, z), 0).$$

By using Fourier transform, we get explicit form of the solution,

$$(9.38) \quad g(t, y, z) = \int_{\mathbb{R}} k(t, \tau, y, z - z') \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz',$$

where

$$k(t, \tau, y, z - z') = \frac{1}{\sqrt{4\pi\varepsilon \int_{\tau}^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))} ds}} \exp\left(-\frac{(z - z')^2}{4\varepsilon \int_{\tau}^t e^{2\varepsilon(\Gamma(t)-\Gamma(s))} ds}\right), \quad \int_{\mathbb{R}} k(t, \tau, y, z) dz = 1$$

$$\Gamma(t) = \int_{\tau}^t \gamma(s, y) ds.$$

We note that,

$$(9.39) \quad \begin{aligned} z^j \partial_z^i g &= \int_{\mathbb{R}} \left(z^j \partial_z^i k(t, \tau, y, z - z') \right) \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz' \\ &= \int_{\mathbb{R}} \left((z^j - z'^j) \partial_z^i k(t, \tau, y, z - z') + (-1)^i z'^j \partial_z^i k(t, \tau, y, z - z') \right) \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz'. \end{aligned}$$

Now, since k is Gaussian,

$$(9.40) \quad \int_{\mathbb{R}} |(z^j - z'^j) \partial_z^i k| dz' \lesssim 1.$$

So, using integration by parts on the 2nd term, we can deduce

$$(9.41) \quad \begin{aligned} \|z^j \partial_z^i g\|_{L^\infty} &\lesssim \|\tilde{\rho}_0\|_{L^\infty} + \left\| \int_{\mathbb{R}} \partial_{z'}^{i-1} k(t, \tau, y, z - z') \{j z'^{j-1} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') + z'^j \partial_{z'} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') e^{-\Gamma(t)}\} dz' \right\|_{L^\infty} \\ &\lesssim \cdots \lesssim \|\tilde{\rho}_0\|_{L^\infty} + \left\| \int_{\mathbb{R}} k(t, \tau, y, z - z') \left\{ \sum_{i_1+i_2=i} (z')^{j-i_1} \partial_{z'}^{i_2} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') e^{-i_2 \Gamma(t)} \right\} dz' \right\|_{L^\infty} \\ &\lesssim \|\tilde{\rho}_0\|_{L^\infty} + \sum_{i_1+i_2=i} \left\| \int_{\mathbb{R}} k(t, \tau, y, z - z') \left\{ (z')^{j-i_1} \partial_{z'}^{i_2} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') \right\} dz' \right\|_{L^\infty}. \end{aligned}$$

By relation of ρ and g , we get

$$(9.42) \quad \|z^j \partial_z^i \rho\|_{L^\infty} \leq \|z^j \partial_z^i \tilde{\rho}\|_{L^\infty} \lesssim \|\tilde{\rho}_0\|_{L^\infty} + \sum_{i_1+i_2=i} \|z^{j-i_1} \partial_z^{i_2} \tilde{\rho}_0\|_{L^\infty}$$

$$\leq \|z^j \partial_z^i \tilde{\rho}\|_{L^\infty} \lesssim \|\rho_0\|_{L^\infty} + \sum_{i_1+i_2=i} \|z^{j-i_1} \partial_z^{i_2} \rho_0\|_{L^\infty}.$$

□

Now we apply Z_3^k to Duhamel's formula to get

$$(9.43) \quad Z_3^k \rho(t) = Z_3^k(S(t, \tau) \rho_0) + \int_0^t Z_3^k(S(t, \tau) G(\tau)) d\tau.$$

Using above Lemma 9.7 twice on the right hand side,

$$(9.44) \quad \begin{aligned} \|Z_3^k \rho(t)\|_{L^\infty} &\lesssim \sum_{i=1}^k \sum_{j=1}^i \left\{ \|\rho_0\|_{L^\infty} + \sum_{j+1+j_2=j} \|z^{j-j_1} \partial_z^{j_2} \rho_0\|_{L^\infty} \right\} \\ &+ \sum_{i=1}^k \sum_{j=1}^i \int_0^t \left\{ \|G\|_{L^\infty} + \sum_{j+1+j_2=j} \|z^{j-j_1} \partial_z^{j_2} G\|_{L^\infty} \right\} d\tau. \end{aligned}$$

Using the fact that ρ (also ρ_0) and G are compactly supported in z , we have

$$\|Z_3^k \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{W^{k,\infty}} + \int_0^t \|G\|_{W^{k,\infty}} d\tau.$$

We also note that other tangential derivatives cases also holds. (This is easier than Z_3 case.) Hence

$$(9.45) \quad \|Z^k \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{Y^{k,0}} + \int_0^t \|G\|_{Y^{k,0}} d\tau.$$

Let's estimate $\|\mathcal{R}\|_{Y^{k,0}}$. Since ρ is compactly supported in z (near the boundary), using Taylor's series and inserting function $\zeta(z) \doteq \frac{z}{1-z}$ (inserting this function is very useful, because existence of $\zeta(z)$ enables us to control using co-normal derivatives of ρ), we have

$$(9.46) \quad \begin{aligned} \|\mathcal{R}\|_{Y^{k,0}} &\leq \|(w_y(t, y, z) - w_y(t, y, 0)) \cdot \nabla_y \rho\|_{Y^{k,0}} + \|(w_3(t, y, z) - z \partial_z w_3(t, y, 0)) \partial_z \rho\|_{Y^{k,0}} \\ &\leq \|\partial_z w_y\|_{Y^{k,0}} \|\zeta(z) \rho\|_{Y^{k+1,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \|\zeta^2(z) \partial_z \rho\|_{Y^{k,0}} \\ &\leq \left(\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \right) \left(\|\zeta(z) \rho\|_{Y^{k+1,0}} + \|\zeta^2(z) \partial_z \rho\|_{Y^{k,0}} \right). \end{aligned}$$

Using anisotropic Sobolev embedding for co-normal derivatives,

$$(9.47) \quad \begin{aligned} \|\zeta(z) \rho\|_{Y^{k+1,0}} &\lesssim \|\zeta(z) \rho\|_{X^{k+1,2}} + \|\partial_z(\zeta(z) \rho)\|_{X^{k+1,1}} \\ &\leq \|\zeta(z) Z^{k+3} \rho\|_{L^2} + \|\partial_z(\zeta(z) Z^{k+2} \rho)\|_{L^2} \\ &\lesssim \|\rho\|_{X^{k+3,0}} + \|\zeta'(z) \rho\|_{X^{k+2,0}} + \|\rho\|_{X^{k+3,0}} \lesssim \|\rho\|_{X^{k+3,0}} \end{aligned}$$

$$(9.48) \quad \|\zeta^2(z) \partial_z \rho\|_{Y^{k,0}} = \|\zeta(z) Z_3 \rho\|_{Y^{k,0}}$$

and $\zeta(z)$ is nice bounded function for all order of derivatives, so at result,

$$(9.49) \quad \|\mathcal{R}\|_{Y^{k,0}} \lesssim \left(\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_{zz} w_3\|_{Y^{k,0}} \right) \|\rho\|_{X^{k+3,0}}.$$

Combining with (9.45), we finish the proof. □

Now, we are ready to get energy estimate for $\|S_n\|_{Y^{k,0}}$.

Proposition 9.8. *Let's define non-dissipation type energy \mathcal{E}_m as*

$$(9.50) \quad \mathcal{E}_{m-2} \doteq \Lambda \left(\frac{1}{c_0}, \|v\|_{X^{m-2,0}} + |h|_{X^{m-1,0}} + \|\partial_z v\|_{X^{m-3,0}} + \|\partial_z v\|_{Y^{\frac{m}{2},0}} \right).$$

(Note that this is equivalent with

$$\mathcal{Q}_{m-2} \doteq \left(\|v\|_{X^{m-2,0}} + |h|_{X^{m-1,0}} + \|S_n\|_{X^{m-3,0}} + \|S_n\|_{Y^{\frac{m}{2},0}} \right).$$

) We have the following estimate for $\|S_n\|_{Y^{\frac{m}{2},0}}$ for sufficiently large $m \geq 9$.

$$(9.51) \quad \|S_n(t)\|_{Y^{\frac{m}{2},0}}^2 \leq \|S_n(0)\|_{Y^{\frac{m}{2},0}}^2 + \varepsilon \int_0^t \mathcal{E}_{m-2} \|\nabla S_n\|_{X^{m-3,0}}^2 d\tau$$

$$\leq \|S_n(0)\|_{Y^{\frac{m}{2},0}}^2 + \varepsilon(\sup_t \mathcal{E}_{m-2})\sqrt{t}\left(\int_0^t \|\nabla S_n\|_{X^{m-3,0}}^2\right)^{1/2}.$$

Proof. We already transformed S_n equation into equivalent- ρ equation system (9.26). From the result of Lemma 9.6, we should estimate the following four terms. Here, $k = \frac{m}{2}$, and m is sufficiently large.

$$\|\rho\|_{X^{k+3,0}}, \|\partial_z w_y\|_{Y^{k,0}}, \|\partial_{zz} w_3\|_{Y^{k,0}}, \|\mathcal{H}\|_{Y^{k,0}}.$$

1), 2) $\|\rho\|_{X^{k+3,0}}$ and $\|\partial_z w_y\|_{Y^{k,0}}$ are trivially controlled by \mathcal{E}_{m-2} by definition of ρ .

3) $\|\partial_{zz} w_3\|_{Y^{k,0}}$: By definition of w ,

$$w = \bar{\chi}(D\Psi)^{-1}(v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi).$$

$$(9.52) \quad \|\partial_{zz} w_3\|_{Y^{k,0}} \leq \|\partial_{zz} (\bar{\chi}(D\Phi)^{-1} \partial_t \Psi)\|_{Y^{k,0}} + \|\partial_{zz} (\bar{\chi}(D\Phi)^{-1} v(t, \Phi^{-1} \circ \Psi))\|_{Y^{k,0}}.$$

For first term,

$$\|\partial_{zz} (\bar{\chi}(D\Phi)^{-1} \partial_t \Psi)\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |h|_{Y^{k,0}} + |\partial_t h|_{Y^{k,0}}\right) \lesssim \mathcal{E}_{m-2}.$$

For second, by using commutator,

$$\begin{aligned} \|\partial_{zz} (\bar{\chi}(D\Psi)^{-1} v_3(t, \Phi^{-1} \circ \Psi))\|_{Y^{k,0}} &\leq \|\bar{\chi} \partial_{zz} ((D\Psi(t, y, 0))^{-1} v(t, \Phi^{-1} \circ \Psi))\|_{Y^{k,0}} + \mathcal{E}_{m-2} \|\partial_z v\|_{Y^{k,0}} \\ &= \|\bar{\chi} \partial_{zz} (v(t, \Phi^{-1} \circ \Psi) \cdot \mathbf{n}^b)\|_{Y^{k,0}} + \mathcal{E}_{m-2}, \end{aligned}$$

where we used

$$((D\Psi(t, y, 0))^{-1} f)_3 = f \cdot \mathbf{n}^b.$$

Main difficulty is two normal derivatives. Meanwhile, by definition, $v(t, \Phi^{-1} \circ \Psi) = u(t, \Psi) \doteq u^\Psi$ and u is divergence free is zero. In the local coordinate, divergence free condition gives,

$$(9.53) \quad \partial_z u^\Psi \cdot \mathbf{n}^b = -\frac{1}{2} \partial_z (\ln|g|) u^\Psi \cdot \mathbf{n}^b - \nabla_{\bar{g}} \cdot u_y^\Psi,$$

which means one normal derivative is replaced by tangential derivative. So,

$$\|\bar{\chi} \partial_{zz} (v(t, \Phi^{-1} \circ \Psi) \cdot \mathbf{n}^b)\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|\partial_z u^\Psi\|_{Y^{k,0}} + |h|_{Y^{k+3,0}}\right) \lesssim \mathcal{E}_{m-2}.$$

In conclusion, $\|\partial_{zz} w_3\|_{Y^{k,0}} \lesssim \mathcal{E}_{m-2}$.

4) $\|\mathcal{H}\|_{Y^{k,0}}$: We have

$$(9.54) \quad \|F_n^\Psi\|_{Y^{k,0}} \leq \|F_n^{\Psi,1}\|_{Y^{k,0}} + \|F_n^{\Psi,2}\|_{Y^{k,0}} + \|\Pi^b F_{S^\times} \mathbf{n}^b\|_{Y^{k,0}}.$$

For first term,

$$\|F_n^{\Psi,1}\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, |\partial_t h|_{Y^{k+1,0}} + |h|_{Y^{k+2,0}} + \|w\|_{Y^{k,0}}\right) \|\partial_z v\|_{Y^{k,0}} \lesssim \mathcal{E}_{m-2}.$$

Second,

$$\|F_n^{\Psi,2}\|_{Y^{k,0}} \lesssim \varepsilon \mathcal{E}_m \|S_n\|_{Y^{k+2,0}} \lesssim \mathcal{E}_m (\varepsilon \|S_n\|_{X^{k+2,0}} + \varepsilon \|\partial_z S_n\|_{X^{k+1,0}}).$$

Note that right hand side can be bounded by dissipation type energy.

Third,

$$\|\Pi^b F_{S^\times} \mathbf{n}^b\|_{Y^{k,0}} \lesssim \mathcal{E}_m \Lambda\left(\frac{1}{c_0}, 1 + \varepsilon \|S_n\|_{Y^{k+2,0}} + \|\Pi^b ((D^\varphi)^2 q) \mathbf{n}^b\|_{Y^{k,0}}\right).$$

$\varepsilon \|S_n\|_{Y^{k+2,0}}$ was treated as second term, and $\|\Pi^b ((D^\varphi)^2 q) \mathbf{n}^b\|_{Y^{k,0}}$ can be estimated since $\Pi^b \partial_z \sim Z_3$.

$$\begin{aligned} \|\Pi^b ((D^\varphi)^2 q) \mathbf{n}^b\|_{Y^{k,0}} &\lesssim \Lambda\left(\frac{1}{c_0}\right) \|\nabla q\|_{Y^{k+1,0}} \\ &\lesssim \mathcal{E}_{m-2} (1 + \varepsilon \|S_n\|_{X^{k+4,0}}). \end{aligned}$$

where last term can be treated similarly as above. (by anisotropic Sobolev embedding) We did all estimate to apply Lemma 9.6 and we get finally

$$(9.55) \quad \|S_n(t)\|_{Y^{\frac{m}{2},0}}^2 \leq \|S_n(0)\|_{Y^{\frac{m}{2},0}}^2 + \varepsilon \int_0^t \mathcal{E}_{m-2} \|\nabla S_n\|_{X^{m-3,0}}.$$

□

10. VORTICITY ESTIMATE

From above section we checked that our energy for $\partial_z v$ is $L^\infty X^{m-3,0}$. However, we should be able to control $X^{m-1,0}$ in space. Using critical idea of [1], we get the regularity of $\partial_z v$ in $L^4 X^{m-1,0}$. We use vorticity to control $\partial_z v$. Note that

$$\|Z^\alpha \partial_z v\| \leq \Lambda_{m,\infty}(h, v)(\|v\|_m + |h|_{m-\frac{1}{2}} + \|\omega\|_{m-1}), \quad |\alpha| = m-1.$$

This means that we are suffice to control vorticity instead of $\partial_z v$. Using vorticity we can neglect pressure effect, however main difficulty in vorticity estimate is that it does not vanish on the free boundary. We first divide the equation of vorticity into homogeneous and non-homogeneous parts and apply critical L^4 in time type estimate from [1]. Applying $\nabla^\varphi \times$ to the Navier-Stokes equation gives,

$$(10.1) \quad \partial_t^\varphi \omega + (v \cdot \nabla^\varphi) \omega - (\omega \cdot \nabla^\varphi) v = \varepsilon \Delta^\varphi \omega.$$

Applying Z^{m-1} gives,

$$(10.2) \quad \partial_t^\varphi Z^{m-1} \omega + (v \cdot \nabla^\varphi) Z^{m-1} \omega - \varepsilon \Delta^\varphi Z^{m-1} \omega = F,$$

where

$$F = Z^{m-1}(\omega \cdot \nabla^\varphi v) + \mathcal{C}_S,$$

where

$$\mathcal{C}_S = \mathcal{C}_S^1 + \mathcal{C}_S^2 + \mathcal{C}_S^3,$$

with

$$\mathcal{C}_S^1 = [Z^{m-1} v_y] \cdot \nabla_y \omega + [Z^{m-1}, V_z] \partial_z \omega \doteq \mathcal{C}_{S_y}^1 + \mathcal{C}_{S_z}^1,$$

$$\mathcal{C}_S^2 = \varepsilon [Z^{m-1}, \Delta^\varphi] \omega,$$

And for boundary data we have, by Lemma 5.5 in [1],

$$(10.3) \quad |(Z^{m-1} \omega)^b| \leq \Lambda_{\infty,6}(|v^b|_{X^{m,0}} + |h|_{X^{m,0}}).$$

Using trace theorem,

$$|(Z^{m-1} \omega)^b| \leq \Lambda_{m,\infty}(h, v) \left(\|\nabla v\|_{X^{m,0}}^{\frac{1}{2}} \|v\|_{X^{m,0}}^{\frac{1}{2}} + \|v\|_{X^{m,0}} + |h|_{X^{m,0}} \right),$$

then the only way to control this boundary terms is,

$$\sqrt{\varepsilon} \int_0^t |(Z^{m-1} \omega)^b|^2 \leq \varepsilon \int_0^t \|\nabla v\|_{X^{m,0}}^2 + \int_0^t \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}}^2 + |h|_{X^{m,0}}^2 \right).$$

It will be useful to keep the term before using young's inequality

$$(10.4) \quad \sqrt{\varepsilon} \int_0^t |(Z^{m-1} \omega)^b|^2 \leq \Lambda_{m,\infty}(h, v) \sqrt{\varepsilon} \left(\|\nabla v\|_{X^{m,0}} \|v\|_{X^{m,0}} + \|v\|_{X^{m,0}}^2 + |h|_{X^{m,0}}^2 \right).$$

We split vorticity as following.

$$Z^{m-1} \omega = \omega_h^{m-1} + \omega_{nh}^{m-1}.$$

I) ω_{nh}^{m-1} solves, nonhomogeneous

$$(10.5) \quad \partial_t^\varphi \omega_{nh}^{m-1} + (v \cdot \nabla^\varphi) \omega_{nh}^{m-1} - \varepsilon \Delta^\varphi \omega_{nh}^{m-1} = F,$$

with initial and zero-boundary condition,

$$(\omega_{nh}^{m-1})^b = 0, \quad (\omega_{nh}^{m-1})_{t=0} = \omega(0).$$

II) ω_h^{m-1} solves, homogeneous

$$(10.6) \quad \partial_t^\varphi \omega_h^{m-1} + (v \cdot \nabla^\varphi) \omega_h^{m-1} - \varepsilon \Delta^\varphi \omega_h^{m-1} = 0.$$

We state the energy estimate for these two vorticity terms.

10.1. Non-homogeneous estimate. We estimate ω_{nh}^α . From propotision 10.2 of [1], by considering time derivative case, we can easily get,

Proposition 10.1. *Non-homogeneous part estimate.*

$$(10.7) \quad \|\omega_{nh}^{m-1}(t)\|^2 + \varepsilon \int_0^t \int_S |\nabla^\varphi \omega_{nh}^{m-1}|^2 dV_t ds \leq \Lambda_0 \left(\|\omega_{v,nh}^{m-1}(0)\|^2 \right) + \varepsilon \Lambda_{m,\infty}(h, v) \int_0^t \|\nabla S_n^v\|_{X^{m-2,0}}^2 ds \\ + \int_0^t \Lambda_{m,\infty}(h, v) \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + \|\omega_v\|_{m-1}^2 + |h|_{X^{m,1}}^2 \right) ds.$$

10.2. Homogeneous estimate. We estimate ω_h^α . We use Theorem 10.6 in [1], and it is trivial to apply it to time-derivatives version.

Proposition 10.2. *Let us assume that*

$$\sup_{[0,t]} \Lambda_{m,\infty}(h, v) + \int_0^t \left(\varepsilon \|\nabla v\|_{X^{m,0}}^2 + \varepsilon \|\nabla S_n\|_{X^{m-2,0}}^2 \right) \leq M.$$

Then we have,

$$(10.8) \quad \left\| \omega_{v,h}^{m-1} \right\|_{L^4(0,T;L^2)}^2 \leq \Lambda(M) \int_0^T \left(\|v\|_{X^{m,0}}^2 + \|\nabla v\|_{X^{m-1,0}}^2 + |h|_m^2 \right) + \frac{\varepsilon}{2} \int_0^T \|\nabla v\|_{X^{m,0}}^2 ds.$$

From above two propositions we know that $\partial_z v \in L^4 X^{m-1,0}$.

11. UNIFORM REGULARITY AND LOCAL EXISTENCE

To get local existence of free-boundary Navier-Stokes equations with viscosity ε , we use local existence theory by A.Tani [7], and combine our propositions to get uniform regularity. This procedure will be very similar to that of [1]. First we fix $m \geq 9$ and viscosity ε , and pick an initial data such that

$$(11.1) \quad I_m(0) \doteq \sum_{k=0}^m \left(\|(\partial_t^k v)(0)\|_{H_{co}^{m-k}} + |(\partial_t^k h)(0)|_{H^{m+1-k}} \right) + |\partial_t^{m-1} \nabla h(0)|_{\frac{3}{2}} \\ + \sum_{k=0}^{m-3} \|(\partial_t^k \nabla v)(0)\|_{H_{co}^{m-3-k}} + \sum_{k=0}^{m/2} \|(\partial_t^k \nabla v)(0)\|_{W_{co}^{\frac{m}{2}-k,\infty}} \leq \infty,$$

with compatibility condition,

$$(11.2) \quad \Pi \left(S^\varphi \partial_t^j v^\varepsilon(0) \right) \mathbf{n} = 0, \quad 0 \leq j \leq m, \quad \Pi \doteq \mathbf{I} - \mathbf{n} \otimes \mathbf{n}.$$

First we regularize v_0^ε by parameter δ , so that $v_0^{\varepsilon,\delta} \in H^{l+1}(S)$, where $l \in (\frac{1}{2}, 1)$. Then by [7], for this initial condition, there is a time interval $T^{\varepsilon,\delta}$ such that on $[0, T^{\varepsilon,\delta}]$, we have a unique solution $v \in W_2^{l+2, \frac{l}{2}+1}([0, T^{\varepsilon,\delta}] \times S) = L^2([0, T^{\varepsilon,\delta}], H^{l+2}(S)) \cap H^{\frac{l}{2}+1}([0, T^{\varepsilon,\delta}], L^2(S))$. Then by parabolic regularity, for $T \in [0, T^{\varepsilon,\delta}]$,

$$(11.3) \quad \Psi_m(T) \doteq \sup_{[0,T]} \left(\|v\|_{X^{m-1,1}}^2 + |h|_{X^{m-1,2}}^2 + \|\partial_z v\|_{X^{m-3,0}}^2 + \|\partial_z v\|_{Y^{\frac{m}{2},0}}^2 \right) + \|\partial_t^m v\|_{L^4 L^2}^2 + (1 + |\partial_t^m \nabla h|_{L^4 L^2}^2) \\ + \|\partial_z v\|_{L^4 X^{m-1,0}}^2 + \varepsilon \int_0^T \int_0^s \|\nabla \partial_t^m v(\tau)\|_{L^2(S)}^2 d\tau ds + \varepsilon \int_0^T \left(\|\nabla v\|_{X^{m-1,1}}^2 + \|\nabla \partial_z v\|_{X^{m-3,0}}^2 \right) < \infty$$

and on the same interval, (by taking $T^{\varepsilon,\delta}$ smaller, if needed.)

$$(11.4) \quad \partial_z \varphi(T) \geq c_0, \quad |h(T)|_{2,\infty} \leq \frac{1}{c_0}.$$

Now let us suppose $\Psi_m(T_0) < \infty$ for some T_0 , then using Stoke's regularity on $[\frac{T_0}{2}, T_0]$, we know that $v(T_0) \in H^{l+1}(S)$, so by considering this as initial condition again, we know that it can be extended to some $T_1 > T_0$. We have to show that this extension is uniform in ε and δ using our propositions.

Instead of $\Psi_m(t)$, we use $\tilde{\Psi}_m(t)$, where

$$(11.5) \quad \tilde{\Psi}_m(T) \doteq \sup_{[0,T]} \left(\|v\|_{X^{m-1,1}}^2 + |h|_{X^{m-1,2}}^2 + \|S_n\|_{X^{m-3,0}}^2 + \|S_n\|_{Y^{\frac{m}{2},0}}^2 \right) + \|\partial_t^m v\|_{L^4 L^2}^2 + (1 + |\partial_t^m \nabla h|_{L^4 L^2}^2)$$

$$+ \|\partial_z v\|_{L^4 X^{m-1,0}}^2 + \varepsilon \int_0^T \int_0^s \|\partial_t^m S_n(\tau)\|_{L^2(S)}^2 d\tau ds + \varepsilon \int_0^T \left(\|S_n\|_{X^{m-1,1}}^2 + \|\nabla S_n\|_{X^{m-3,0}}^2 \right) < \infty.$$

(In fact, all our propositions were written in terms of $\tilde{\Psi}$.) These two Ψ_m and $\tilde{\Psi}_m$ are equivalent as explained in Lemma 8.2.(the opposite direction is trivial.)

To derive uniform time interval, we choose R and c_0 so that, $\frac{1}{c_0} \ll R$, and define,

$$(11.6) \quad T_*^{\varepsilon, \delta} = \sup \left\{ T \in [0, 1] \text{ s.t. } \tilde{\Psi}_m(t) \leq R, \quad |h(t)|_{2, \infty} \leq \frac{1}{c_0}, \quad \partial_z \varphi(t) \geq c_0, \quad \forall t \in [0, T] \right\}.$$

Suppose $\tilde{\Psi}_m(T) \leq R$ then, now we combine proposition 5.4, 6.4, 7.2, 8.4, 9.8, and 10.2. (and we denote $\Lambda\left(\frac{1}{c_0}, \cdot\right) = \Lambda_0(\cdot)$) First let us consider h -regularity problem. On the right hand side of proposition 5.4, we use proposition 7.2 to $\int_0^t |h|_{X^{m-1, \frac{5}{2}}}^2$ to change this into $\int_0^t |\partial_t \nabla h|_{X^{m-1,0}}^2$, i.e $\partial_t \sim \partial_x^{3/2}$. Note that we got L^4 type energy estimate to get $\int_0^t \|\partial_t^{m-1} \nabla q\|^2$ on the right hand side of proposition 6.4. This will generate $\|\nabla h\|_{X^{m-1, \frac{3}{2}}}^2$ and we use proposition 7.2 again to get $\int_0^t \|\partial_t \nabla h\|_{X^{m-1,0}}^2$. Since we have L_t^4 energy, we can control this term. Also we mention that when we use (7.2), $\bar{\theta} |Z^{m-1} h_t(t)|_{L^2(\partial S)}^2 + \sqrt{t} |Z^{m-1} \nabla h_t(t)|_{L^2(\partial S)}^2$ is absorbed by energies of the next step energy when $Z^{m-1} \neq \partial_t^{m-1}$ for sufficiently small θ and t . For $Z^{m-1} = \partial_t^{m-1}$, next step energy is not L^∞ type, so we cannot make it absorbed directly. Instead, from initial condition, $\|\partial_t^m h(0)\|_{H^1}$ is finite. So using continuity argument, we can choose sufficiently small $\bar{\theta}$ and t depending on initial data so that $\bar{\theta} \|\partial_t^m h(t)\|_{L^2}$ is small.(say $\leq \frac{1}{2}$) For about $\partial_z v$ problem, it is resolved by proposition 8.4, 9.8, and 10.2. Note that it has nothing to do with All-time derivatives estimate. After all, by adding 5.4, 6.4, 8.4, 9.8, and 10.2, and considering equivalences among $\partial_z v$, S_n , and ω , we can derive

$$(11.7) \quad \Psi_m(T) \leq (1+T)\Lambda\left(\frac{1}{c_0}, I_m(0)\right) + \Lambda_0(R)T^{1/2} \leq \Lambda\left(\frac{1}{c_0}, I_m(0)\right) + \Lambda_0(R)T^{1/2}.$$

Now, we calculate conditions in (11.6).

$$(11.8) \quad |h(t)|_{2, \infty} \leq |h(0)|_{2, \infty} + \Lambda_0(R)T \quad \forall t \in [0, T],$$

and since we've chosen A in diffeomorphism to be 1, at initial time,

$$(11.9) \quad \partial_z \varphi(t) \geq 1 - \int_0^t \|\partial_t \nabla \eta\|_{L^\infty} \geq 1 - \Lambda_0(R)T, \quad \forall t \in [0, T].$$

From (11.7), (11.8), and (11.9), we see that right hand side is independent to ε and δ , so are possible to choose $R = \Lambda\left(|h_0|_{2, \infty}, I_m(0)\right)$ which satisfies that there exist T_* (independent to ε, δ) such that $\forall t \in [0, T_*]$,

$$(11.10) \quad \Psi_m(t) \leq \frac{R}{2}, \quad |h(t)|_{2, \infty} \leq \frac{1}{2c_0}, \quad \partial_z \varphi(t) \geq c_0 + \frac{1-c_0}{2} > c_0.$$

This implies $T_* < T_*^{\varepsilon, \delta}$, hence T_* implies there exist uniform time, independent to ε, δ . Since $\Psi_m(T_*)$ is uniformly bounded in δ , we can pass the limit, $\delta \rightarrow 0$, by strong compactness argument. Before finishing local existence section, we note about compatibility condition. Since our solution space include ∂_t^j , $j \leq m$, our initial condition should have information about $\partial_t^j v^\varepsilon(0)$ and $\partial_t^j h^\varepsilon(0)$. These satisfy compatibility condition for Stress-continuity condition, so $\left(S^\varphi \partial_t^j v^\varepsilon(0)\right) \mathbf{n}$ must not have tangential part,

$$(11.11) \quad \Pi\left(S^\varphi \partial_t^j v^\varepsilon(0)\right) \mathbf{n} = 0.$$

12. UNIQUENESS

12.1. Uniqueness for Navier-Stokes. We prove uniqueness of Theorem 1.3. As usual, we consider two solution sets $(v_1^\varepsilon, \varphi_1^\varepsilon, q_1^\varepsilon), (v_2^\varepsilon, \varphi_2^\varepsilon, q_2^\varepsilon)$ with same initial condition and proper compatibility conditions. Then on the interval $[0, T^\varepsilon]$, we have uniform bounds of energy,

$$\Psi_m^i(T^\varepsilon) \leq R, \quad i = 1, 2.$$

Let,

$$(12.1) \quad \bar{v}^\varepsilon \doteq v_1^\varepsilon - v_2^\varepsilon, \quad \bar{h}^\varepsilon \doteq h_1^\varepsilon - h_2^\varepsilon, \quad \bar{q}^\varepsilon \doteq q_1^\varepsilon - q_2^\varepsilon.$$

We will make system of equations for $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$ and do energy estimate. By divergence free condition, $\nabla^{\varphi_i} \cdot v_i^\varepsilon = 0$,

$$\left(\partial_t + v_{y,i}^\varepsilon \cdot \nabla_y + V_{z,i}^\varepsilon \partial_z \right) v_i^\varepsilon + \nabla^{\varphi_i} q_i^\varepsilon - \varepsilon \Delta^{\varphi_i} v_i^\varepsilon = 0.$$

Then we have equation about $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$. First for Navier-Stokes,

$$(12.2) \quad \left(\partial_t + v_{y,1}^\varepsilon \cdot \nabla_y + V_{z,1}^\varepsilon \partial_z \right) \bar{v}^\varepsilon + \nabla^{\varphi_1} \bar{q}^\varepsilon - \varepsilon \Delta^{\varphi_1} \bar{v}^\varepsilon = F,$$

where

$$\begin{aligned} F = & (v_{y,2}^\varepsilon - v_{y,1}^\varepsilon) \cdot \nabla_y v_2^\varepsilon + (V_{z,2}^\varepsilon - V_{z,1}^\varepsilon) \partial_z v_2^\varepsilon - \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) (P_1^* \nabla q_2^\varepsilon) + \frac{1}{\partial_z \varphi_2^\varepsilon} ((P_2 - P_1)^* \nabla q_2^\varepsilon) \\ & + \varepsilon \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) \nabla \cdot (E_1 \nabla v_2^\varepsilon) + \varepsilon \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot ((E_2 - E_1) \nabla v_2^\varepsilon). \end{aligned}$$

For divergence-free condition,

$$(12.3) \quad \nabla^{\varphi_1} \cdot \bar{v}^\varepsilon = - \left(\frac{1}{\partial_z \varphi_2^\varepsilon} - \frac{1}{\partial_z \varphi_1^\varepsilon} \right) \nabla \cdot (P_1 v_2^\varepsilon) - \frac{1}{\partial_z \varphi_2^\varepsilon} \nabla \cdot ((P_2 - P_1) v_2^\varepsilon).$$

For Kinematic boundary condition,

$$(12.4) \quad \partial_t \bar{h}^\varepsilon - (v_{y,1}^\varepsilon)^b \cdot \nabla h + \left((v_{z,1}^\varepsilon)^b - (v_{z,2}^\varepsilon)^b \right) = - \left((v_{y,2}^\varepsilon)^b - (v_{y,1}^\varepsilon)^b \right) \cdot \nabla h_2^\varepsilon.$$

Continuity of stress tensor condition becomes,

$$\begin{aligned} (12.5) \quad & \bar{q}^\varepsilon \mathbf{n}_1 - 2\varepsilon (S^{\varepsilon_1} \bar{v}^\varepsilon) \mathbf{n}_1 = g \bar{h}^\varepsilon - \eta \nabla \cdot \left(\frac{\nabla \bar{h}^\varepsilon}{\sqrt{1 + |\nabla h_1^\varepsilon|^2}} \right) \\ & + 2\varepsilon \left((S^{\varphi_1} - S^{\varphi_2}) v_2^\varepsilon \right) \mathbf{n}_1 + 2\varepsilon (S^{\varphi_2} v_2^\varepsilon) (\mathbf{n}_1 - \mathbf{n}_2) - \eta \nabla \cdot \left(\left\{ \frac{1}{\sqrt{1 + |\nabla h_1^\varepsilon|^2}} - \frac{1}{\sqrt{1 + |\nabla h_2^\varepsilon|^2}} \right\} \nabla h_2^\varepsilon \right). \end{aligned}$$

Using above 4 equations, we get L^2 - energy estimate, (since initial condition is zero, no initial term appear on right hand side)

$$(12.6) \quad \|\bar{v}^\varepsilon(t)\|_{L^2}^2 + |\bar{h}^\varepsilon(t)|_{H^1}^2 + \varepsilon \int_0^t \|\nabla \bar{v}^\varepsilon\|_{L^2}^2 ds \leq \Lambda(R) \int_0^t \left(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + |\bar{h}^\varepsilon(s)|_{H^{\frac{3}{2}}}^2 \right) ds.$$

We skip detail calculation, since we can use our previous energy estimates basically. But, in above equations for $(\bar{v}^\varepsilon, \bar{h}^\varepsilon, \bar{q}^\varepsilon)$, right hand side does not have low order than L^2 energy. However we have uniform bound of m-order energy, so we can extract bad high order terms into $\Lambda(R)$. To estimate $|\bar{h}^\varepsilon(s)|_{H^{\frac{3}{2}}}$, we don't have to take time derivatives as like in previous section, since we already have bounded high order energy. And moreover, we don't need uniform estimate in ε , since we are dealing about for fixed ε . So, we estimate $\varepsilon |\bar{h}^\varepsilon(s)|_{H^{\frac{3}{2}}}^2$.

Lemma 12.1. *For every $m \in \mathbb{N}$, $\varepsilon \in (0, 1)$, we have the estimate*

$$(12.7) \quad \varepsilon |h(t)|_{m+\frac{1}{2}}^2 \leq \varepsilon |h(0)|_{m+\frac{1}{2}}^2 + \varepsilon \int_0^t |v^b|_{m+\frac{1}{2}}^2 + \int_0^t \Lambda_{1,\infty} \left(\|v\|_m^2 + \varepsilon |h|_{m+\frac{1}{2}}^2 \right) ds,$$

where

$$\Lambda_{1,\infty} = \Lambda \left(|\nabla h|_{L^\infty} + \|v\|_{1,\infty} \right).$$

Proof. See proposition 3.4 in [1]. □

This is true for our case, since it comes from Kinematic boundary condition. We can also apply this lemma, to \bar{h}^ε case, (surely, $\bar{h}^\varepsilon(0) = 0$) and then combine with above L^2 estimate, then we get the following estimate. (non-uniform in ε)

$$(12.8) \quad \|\bar{v}^\varepsilon(t)\|_{L^2}^2 + |\bar{h}^\varepsilon(t)|_{H^1}^2 + \varepsilon |\bar{h}^\varepsilon(t)|_{H^{\frac{3}{2}}}^2 + \varepsilon \int_0^t \|\nabla \bar{v}^\varepsilon\|_{L^2}^2 ds \leq \frac{\Lambda(R)}{\varepsilon} \int_0^t \left(\|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \varepsilon |\bar{h}^\varepsilon(s)|_{H^{\frac{3}{2}}}^2 \right) ds.$$

Then we can use Gronwall's inequality to get uniqueness. So finish uniqueness part of theorem 1.2.

12.2. Uniqueness for Euler. Since our estimate in above subsection (uniqueness for Navier-Stokes) is not uniform in ε , result cannot be applied to Euler equation. As like in Navier-Stokes case, let we have two solutions $(v_1, h_1, q_1), (v_2, h_2, q_2)$ with same initial condition. Suppose,

$$(12.9) \quad \sup_{[0,T]} \left(\|v_i\|_m + \|\partial_z v_i\|_{m-1} + \|\partial_z v_i\|_{\frac{m}{2},\infty} + |h_i|_{m+1} \right) \leq R, \quad i = 1, 2.$$

(This is true from result in Theorem 1.2) Define $\bar{v} \doteq v_1 - v_2$, $\bar{h} \doteq h_1 - h_2$, $\bar{q} \doteq q_1 - q_2$ and we write equation of $(\bar{v}, \bar{h}, \bar{q})$, as before. Euler equation becomes,

$$(12.10) \quad \left(\partial_t + v_{y,1} \cdot \nabla_y + V_{z,1} \partial_z \right) \bar{v} + \nabla^{\varphi_1} \bar{q} = F',$$

where

$$F' = (v_{y,2} - v_{y,1}) \cdot \nabla_y v_2 + (V_{z,2} - V_{z,1}) \partial_z v_2 - \left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \right) (P_1^* \nabla q_2) + \frac{1}{\partial_z \varphi_2} ((P_2 - P_1)^* \nabla q_2).$$

For divergence-free condition,

$$(12.11) \quad \nabla^{\varphi_1} \cdot \bar{v} = - \left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1} \right) \nabla \cdot (P_1 v_2) - \frac{1}{\partial_z \varphi_2} \nabla \cdot ((P_2 - P_1) v_2).$$

For Kinematic boundary condition,

$$(12.12) \quad \partial_t \bar{h} - v_{y,1}^b \cdot \nabla h + (v_{z,1}^b - v_{z,2}^b) = - (v_{y,2}^b - v_{y,1}^b) \cdot \nabla h_2.$$

Continuity of stress tensor condition becomes,

$$(12.13) \quad \bar{q} \mathbf{n}_1 = g \bar{h} - \eta \nabla \cdot \left(\frac{\nabla \bar{h}}{\sqrt{1 + |\nabla h_1|^2}} \right) - \eta \nabla \cdot \left(\left\{ \frac{1}{\sqrt{1 + |\nabla h_1|^2}} - \frac{1}{\sqrt{1 + |\nabla h_2|^2}} \right\} \nabla h_2 \right).$$

By performing basic L^2 -estimate, as similarly above, (we skip detail here)

$$(12.14) \quad \|\bar{v}(t)\|_{L^2}^2 + |\bar{h}(t)|_{H^1}^2 \leq \Lambda(R) \int_0^t \left(\|\bar{v}(s)\|_{H^1}^2 + |\bar{h}(s)|_{H^{\frac{3}{2}}}^2 \right) ds.$$

We should control $\|v\|_1$ on right hand side. But, since there are no dissipation on left hand side, we cannot make it absorbed. Instead, we use vorticity. Let's define vorticity $\omega = \nabla^\varphi \times v$ (which is equivalent to $\omega = (\nabla \times u)(t, \Phi)$). We have

$$\begin{aligned} \omega \times \mathbf{n} &= \frac{1}{2} (D^\varphi v \mathbf{n} - (D^\varphi v)^T \mathbf{n}) \\ &= S^\varphi v \mathbf{n} - (D^\varphi v)^T \mathbf{n} = \frac{1}{2} \partial_{\mathbf{n}} u - g^{ij} (\partial_j v \cdot \mathbf{n}) \partial_{y^i}. \end{aligned}$$

Hence, it is suffice to estimate ω instead of $\partial_z v$, *i.e*

$$(12.15) \quad \|\partial_z v\|_{L^2} \leq \Lambda(R) \left(\|\omega\|_{L^2} + \|v\|_1 + |h|_{\frac{3}{2}} \right).$$

To estimate ω , we use vorticity equation.

$$(12.16) \quad \left(\partial_t^{\varphi_i} + v_i \cdot \nabla^{\varphi_i} \right) \omega_i = (\omega_i \cdot \nabla^{\varphi_i}) v_i.$$

L^2 energy estimate of $\bar{\omega}$ is

$$(12.17) \quad \|\bar{\omega}(t)\|_{L^2}^2 \leq \Lambda(R) \int_0^t \left(|\bar{h}(s)|_1^2 + \|\bar{v}(s)\|_1^2 + \|\partial_z \bar{v}(s)\|_{L^2}^2 + \|\bar{\omega}(s)\|_{L^2}^2 \right) ds.$$

We also should control $|h|_{L^2 H^{\frac{3}{2}}}^2$. As similar to Dirichlet-Neumann estimate we can control this by $|\partial_t h|_{L^2 L^2}^2$. And, from kinematic boundary condition of \bar{h} , we easily get

$$(12.18) \quad |\partial_t \bar{h}(t)|_{L^2}^2 \leq \Lambda(R) \left(\|\nabla \bar{v}\|_{L^2}^2 + |\bar{h}|_{H^1}^2 \right).$$

Then we can use Gronwall's inequality to get uniqueness. So finish uniqueness part of theorem 1.2 and theorem 1.3.

13. INVISCID LIMIT

In this section we send ε to zero, and get a unique solution of free boundary Euler equation. For $\varepsilon \in (0, 1]$ and $T \leq T_*$, we have uniform energy bound,

$$(13.1) \quad \begin{aligned} \Psi_m(T) \doteq & \sup_{[0, T]} \left(\|v^\varepsilon\|_{X^{m-1,1}}^2 + |h^\varepsilon|_{X^{m-1,2}}^2 + \|\partial_z v^\varepsilon\|_{X^{m-3,0}}^2 + \|\partial_z v^\varepsilon\|_{Y^{\frac{m}{2},0}}^2 \right) \\ & + \|\partial_t^m v^\varepsilon\|_{L^4 L^2}^2 + 1 + |\partial_t^m \nabla h^\varepsilon|_{L^4 L^2}^2 + \|\partial_z v^\varepsilon\|_{L^4 X^{m-1,0}}^2 \\ & + \varepsilon \int_0^T \int_0^s \|\nabla \partial_t^m v^\varepsilon(\tau)\|_{L^2(S)}^2 d\tau ds + \varepsilon \int_0^T \left(\|\nabla v^\varepsilon\|_{X^{m-1,1}}^2 + \|\nabla \partial_z v^\varepsilon\|_{X^{m-3,0}}^2 \right) < \infty. \end{aligned}$$

So we have uniform boundness for v^ε in $L^\infty([0, T], H_{co}^m)$ and h^ε in $L^\infty([0, T], H^{m+1})$. And, by Rellich-Kondrachov theorem, we have, for each t , compactness of $v^\varepsilon(t)$ in $H_{co,loc}^{m-1}$ and $h^\varepsilon(t)$ in H_{loc}^m . And from our energy function, we have a uniform boundness of $\partial_t v^\varepsilon(t)$ in $H_{co,loc}^{m-1}$ and of $\partial_t h^\varepsilon(t)$ in H_{loc}^m for $\forall t \in [0, T]$. Now, we have subsequence v^{ε_n} , h^{ε_n} , such that

$$(13.2) \quad \begin{aligned} v^{\varepsilon_n} & \rightarrow v, \text{ strongly in } C([0, T], H_{co,loc}^{m-1}), \\ h^{\varepsilon_n} & \rightarrow h, \text{ strongly in } C([0, T], H_{loc}^m). \end{aligned}$$

About pressure, from pressure section, we have boundness of ∇q^ε in $L^2([0, T] \times S)$, so get some q such that,

$$\nabla q^{\varepsilon_n} \rightharpoonup \nabla q, \text{ weakly in } L^2([0, T] \times S)$$

and limit functions $(v, h, \nabla q)$ satisfy

$$(13.3) \quad \sup_{[0, T]} \left(\|v\|_{H_{co}^m}^2 + |h|_{H^{m+1}}^2 + \|\partial_z v\|_{H_{co}^{m-1}}^2 + \|\partial_z v\|_{W_{co}^{\frac{m}{2}, \infty}}^2 \right) \leq R,$$

$$(13.4) \quad \begin{aligned} \Psi_m(T) \doteq & \sup_{[0, T]} \left(\|v\|_{X^{m-1,1}}^2 + |h|_{X^{m-1,2}}^2 + \|\partial_z v\|_{X^{m-3,0}}^2 + \|\partial_z v\|_{Y^{\frac{m}{2},0}}^2 \right) \\ & + \|\partial_t^m v\|_{L^4 L^2}^2 + 1 + |\partial_t^m \nabla h|_{L^4 L^2}^2 + \|\partial_z v\|_{L^4 X^{m-1,0}}^2 < R. \end{aligned}$$

Now, we can pass to the limit and get the fact that $(v, h, \nabla q)$ is a weak solution of Euler equation.(interior). For boundary condition, first we can assume that the trace(boundary function),

$$(13.5) \quad v^{\varepsilon_n}(t, y, z=0) \rightharpoonup v^b, \text{ weakly in } L^2([0, T] \times S)$$

for some v^b . In kinematic boundary condition, v^b is linear and we have strong convergence of h , so kinematic boundary condition is satisfied weakly surely. Next, for continuity of stress tensor condition, by bounded lipschitz norm of v^{ε_n} , $2\varepsilon(Su)\mathbf{n} \rightarrow 0$ in weak limit process. And, limit of surface tension part is trivial by strong convergence of h . Hence, in the weak sense,

$$(13.6) \quad q^b = gh - \eta \nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right).$$

Hence we have a weak solution (v, h) which is strong $H_{co,loc}^{m-1} \times H_{co,loc}^m$ -convergence of $(v^{\varepsilon_n}, h^{\varepsilon_n})$.(and global for weak convergence in $L^2 \times H^1$) Moreover this limit is unique by previous section. Meanwhile, we can get strong convergence (non-local) in $L^2 \times H^1$. To get this, we just investigate norm convergence. For $t \in [0, T]$, using basic L^2 -energy estimate and uniform boundness of high order energy,

$$(13.7) \quad \begin{aligned} & \left(\|v^{\varepsilon_n} J^{\varepsilon_n}(t)\|_{L^2}^2 + g|h^{\varepsilon_n}(t)|_{L^2}^2 + 2\eta|\sqrt{1 + |\nabla h^{\varepsilon_n}(t)|^2} - 1|_{L^1} \right) \\ & - \left(\|v_0^{\varepsilon_n} J_0^{\varepsilon_n}\|_{L^2}^2 + g|h_0^{\varepsilon_n}|_{L^2}^2 + 2\eta|\sqrt{1 + |\nabla h_0^{\varepsilon_n}|^2} - 1|_{L^1} \right) \leq \varepsilon_n \Lambda(R) \rightarrow 0, \text{ as } \varepsilon_n \rightarrow 0, \end{aligned}$$

where $J^\varepsilon \doteq (\partial_z \varphi^\varepsilon)^{1/2}$ and ε_n on the right hand side come from dissipation of energy estimate. We assume that $\|v_0^\varepsilon - v_0\|_{L^2} \rightarrow 0$, $\|h_0^\varepsilon - h_0\|_{H^1} \rightarrow 0$ in statement of theorem 1.3 and

$$\|\partial_z \varphi^\varepsilon|_{t=0} - \partial_z \varphi|_{t=0}\|_{L^2} \lesssim |h_0^\varepsilon - h_0|_{H^{\frac{1}{2}}} \leq \Lambda(R) |h_0^\varepsilon - h_0|_{L^2}^{\frac{1}{2}}.$$

This implies

$$(13.8) \quad \lim_{\varepsilon \rightarrow 0} \left(\|v_0^{\varepsilon_n} J_0^{\varepsilon_n}\|_{L^2}^2 + g|h_0^{\varepsilon_n}|_{L^2}^2 + 2\eta|\sqrt{1 + |\nabla h_0^{\varepsilon_n}|^2} - 1|_{L^1} \right) = \|v_0 J_0\|_{L^2}^2 + g|h_0|_{L^2}^2 + 2\eta|\sqrt{1 + |\nabla h_0|^2} - 1|_{L^1}.$$

Lastly, using energy conservation in Euler equation ($\varepsilon = 0$, in basic L^2 -estimate), we get

$$(13.9) \quad \|v_0 J_0\|_{L^2}^2 + g|h_0|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h_0|^2} - 1|_{L^1} = \|vJ\|_{L^2}^2 + g|h|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h|^2} - 1|_{L^1}.$$

Finally we get norm convergence.

$$(13.10) \quad \lim_{\varepsilon \rightarrow 0} \left(\|v^{\varepsilon_n} J^{\varepsilon_n}(t)\|_{L^2}^2 + g|h^{\varepsilon_n}(t)|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h^{\varepsilon_n}(t)|^2} - 1|_{L^1} \right) = \|vJ\|_{L^2}^2 + g|h|_{L^2}^2 + 2\eta|\sqrt{1+|\nabla h|^2} - 1|_{L^1}.$$

With weak convergence, this implies strong convergence to (vJ, h) in $L^2 \times H^1$. And strong convergence of h means $(v^\varepsilon, h^\varepsilon) \rightarrow (v, h)$ strongly in $L^2 \times H^1$. (without J) L^∞ -type convergence can be done by L^2 convergence, uniform energy boundness, and anisotropic embedding.

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REFERENCES

- [1] N.Masmoudi and F.Rousset, *Uniform regularity and vanishing viscosity limit for the free surface navier-stokes equations*, arXiv:1202.0657.
- [2] B. Alvarez-Samaniego, D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math, 171 (2008), 485-541.
- [3] D.Lannes, *Well-posedness for the water-waves equations in \mathbb{R}^2* , J.Amer.Math.Soc., 18(3):605-654, 2005
- [4] B.Schweizer, *On the three-dimensional Euler equations with a free boundary subject to surface tension*, Ann.Inst.H.Poincaré Anal.Non Linéaire, 22(6):753-781, 2005.
- [5] J.T.Beale, *The initial value problem for the Navier-Stokes equations with a free surface.*, Comm. Pure. Appl. Math, 34(3):359-392, 1981
- [6] Allain.G, *Small-time existence for the Navier-Stokes equations with a free surface.*, Appl.Math.Optim, 16:37-50, 1987
- [7] A.Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface.*, Arch.Rational Mech.Anal, 133(4):299-331, 1996
- [8] J.T.Beale, *Large-time regularity for viscous surface waves.*, Arch.Rational Mech.Anal, 84(4):307-352, 1983/84
- [9] S.Wu *Well-Posedness in Sobolev spaces of the full water wave problem in 2-D.*, Invent. Math, 130:39-72, 1997
- [10] S.Wu *Well-Posedness in Sobolev spaces of the full water wave problem in 3-D.*, J.Amer.Math.Soc, 12:445-495, 1999
- [11] S.Wu *Almost global well-posedness for the 2-D full water wave problem*, Invent. Math, 177, no.1, 45135, 2009
- [12] P.Germain, N.Masmoudi, and J.Shatah *Global solutions for the gravity water waves equation in dimension 3.*, C.R.Math.Acad.Sci.Paris, 347(15-16):897-902, 2009
- [13] D.Christodoulou and H.Lindblad *On the motion of the free surface of a liquid.*, Comm.Pure.Appl.Math, 53(12):1536-1602, 2000
- [14] H.Lindblad *Well-posedness for the motion of an incompressible liquid with free surface boundary*, Ann.of Math.(2), 162(1):109-194, 2005
- [15] N.Masmoudi and F.Rousset, *Uniform regularity for the Navier-Stokes equation with Navier boundary condition*, Arch.Rational Mech.Anal (to appear)
- [16] D. Coutand and S. Shkoller *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., 20: 829930, 2007.
- [17] J. Shatah and C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation.*, Comm. Pure Appl. Math., 61(5):698744, 2008.
- [18] Yanjin Wang and Zhouping Xin, *Vanishing viscosity and surface tension limits of incompressible viscous surface waves.*, arXiv:1504.00152 [math.AP]

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